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Solving system of linear differential equations using haar wavelet

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Abstract: *In this paper, we present an approximate numerical solution of system of linear differential equations using Haar wavelet method. Haar wavelet method is used because its computation is simple as it converts the problem into algebraic matrix equation. The results and graphs show that the proposed way is quite reasonable when compared to exact solution.*

Keywords: *Haar wavelet;System of linear differential equations; Matlab.*

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1 Introduction

In the recent years wavelet approach has become more popular in the field of numerical approximations. Different types of wavelets and approximating functions have been used in numerical solution of initial and boundary value problems. Chen and Hsiao [2] have gained popularity, due to their useful contribution in wavelet. Lepik [8 - 10] applied Haar wavelet in solving differential equations and partial differential equations. According to Lapik, the unknown function is defined according to the higher order derivative and then by using the integrations of the Haar wavelet it is possible to obtain some algebraic systems in the unknown wavelet coefficients. When these systems are solved, we obtain the coefficients of some Haar wavelet series and this series gives us the wavelet solution. Lapik also showed that the approximation is quite good already with few coefficients. Moreover, Lepik in some

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further papers gave also the solution of integral equations and integro-differential equations by using this method which is based on the operational matrices defined by him and previously by Chen-Hsiao. G. Hariharan et. al. [7] also gave a simple method for solution of partial differential equations. Vedat Suat Erturk and Shather Momani [4] solve system of fractional differential equations using differential transform method. Sachin Bhalekar and Varsha Daftardar – Gejji [1] gave a new iterative method for system of nonlinear functional equations.

In the present paper, a new direct computational method for solving system of linear differential equations is introduced. This method consists of reducing the problem to a set of algebraic equations by first expanding the terms, which has maximum derivative, given in the equation as Haar functions with unknown coefficients. The differentiation of Haar wavelet always results in impulse functions which must be avoided, in the procedure, the integration of Haar wavelet is preferred. Since the integration of the Haar functions vector is continuous function, the solutions obtained are continuous.
Linear differential equations play very important role in modeling numerous problems in physics, chemistry, Biology and Engineering science [5, 6]. Many problems can be modeled as system of linear differential equations, integral equations, fractional differential equations, partial differential equations. Since some system of linear differential equations do not have exact solution, numerical methods are widely used to solve these equations. Several techniques such as Adomain decomposition method [5], Variational iteration method, homotopy perturbation method have been used for solving these problems. Most of these techniques encounter a considerable size of difficulty. But our method is new and very easy to use. One main advantage of this method is that, we don't need to solve it manually it is fully computer supported.

2 Haar wavelet:

The Haar wavelet was first introduced by Alfred Haar [6] in 1910. Haar wavelet is a certain sequence of rescaled "square-shaped" function which together forms a wavelet family or basis. Haar wavelet is defined as $t \in [0\ 1]$

$$\psi(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t \leq 1 \\ 0 & otherwise \end{cases} \tag{1}$$

Haar wavelet family for $t \in [0\ 1]$ is defied as

$$h_i(t) = \begin{cases} 1 & \text{for} \quad t \in [\eta_1,\ \eta_2) \\ -1 & \text{for} \quad t \in [\eta_2,\ \eta_3] \\ 0 & \quad otherwise \end{cases} \tag{2}$$

Where $\eta_1 = \frac{K}{m}$, $\eta_2 = \frac{K+0.5}{m}$, $\eta_3 = \frac{K+1}{m}$. The integer m $= 2^j (j = 0, 1, \ldots, J)$ indicates the level of the wavelet; $k = 0, 1, ..., m-1$ is the translation parameter. The maximal level of relation is J. The index i is calculated according to the formula $i = m+k+1$; In the case of minimal values $m = 1, k = 0$, we have $i = 2$. The maximum value of i is $i = 2^{J+1} = M$. It is assume that the value $i = 1$ corresponding to the scaling function for which $h_1 = 1$ for $t \in [0\ 1]$.
By Hsiao-Chen method [3]

$$p_{i,1}(t) = \int_0^x h_i\,(t)\,dx, \tag{3}$$

$$p_{i,v}\,(t) = \int_0^x p_{i,v-1}\,(t)\,dx, \quad v = 2, 3 \ldots \tag{4}$$

Carrying out these integrations with the aid equation (3), we have

$$p_{i,1}\,(x) = \begin{cases} t - \eta_1 & \text{for} \quad t \in [\eta_1,\ \eta_2) \\ \eta_3 - t & \text{for} \quad x \in [\eta_2,\ \eta_3] \\ & \qquad\quad elsewhere \end{cases} \tag{5}$$

$$p_{i,2}\,(t) = \begin{cases} \frac{1}{2}(t-\eta_1)^2 & \text{for} \quad t \in [\eta_1,\ \eta_2) \\ \frac{1}{4m^2} - \frac{1}{2}(\eta_3 - t)^2 & \text{for} \quad t \in [\eta_2,\ \eta_3] \\ \frac{1}{4m^2} & \text{for} \quad t \in [\eta_3,\ \ 1] \\ 0 & \qquad\quad elsewhere \end{cases} \tag{6}$$

Any function $y(t) \in L^2([0,1])$ can be expanded in Haar series $y(t) = \sum_{i=1}^{8} a_i h_i(t)$. where $a_i,\ i =\ 1,\ 2\ldots$ is the Haar coefficient, which is given by

$$a_i =\ 2^j \int_0^1 y(t) h_i(t) dt.$$

Which are determined such that the following integral square error ε is minimized

$$\varepsilon = \int_0^1 \left[\mathrm{y}\,(\mathrm{t}) - \sum_{i=1}^{m} a_i h_i(t) \right]^2 dt, \quad m = 2^j, \quad j \in \{0\} \cup N$$

The series expansion of $y(t)$ contains an infinite terms. If $y(t)$ is piecewise constant, or may be approximated as piecewise constant during each subinterval, then $y(t)$ will be terminated at finite terms, i.e.

$$y(t) \approx \sum_{i=1}^{m} a_i h_i(t) = a^T H$$

Where $m = 2^j$, the superscript T indicates transposition. The Haar coefficient vector a^T and Haar function vector H are defined as

$$a^T = [a_1, a_1, a_2, \ldots\ a_m]$$

$$H = \begin{bmatrix} h_1^T \\ h_2^T \\ \vdots \\ h_m^T \end{bmatrix} = \begin{bmatrix} h_{1,0} & h_{1,1} & \cdots & h_{1,m} \\ h_{2,0} & h_{2,0} & \cdots & h_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ h_{m,0} & h_{m,1} & \cdots & h_{m,m} \end{bmatrix}$$

Where $h_1^T,\ h_2^T \ldots h_m^T$ are the discrete form of the Haar wavelet.
Haar wavelet for m = 8 is given by

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

3 Method for solutions of system of linear differential equations:

In this study, we consider following system of linear differential equations

$$\begin{aligned} D^{\alpha_1} y_1(t) &= f_1(t,\ y_1, y_2 \ldots y_n) \\ D^{\alpha_2} y_2(t) &= f_2(t,\ y_1, y_2 \ldots y_n) \\ &\vdots \\ &\vdots \\ D^{\alpha_n} y_n(t) &= f_n(t,\ y_1, y_2 \ldots y_n) \end{aligned}$$

Where D^{α_j} is the derivative of y_i of order α_j, $0 < \alpha_j \leq 1$, subject to the initial conditions $y_1(0) = c_1,\ \ y_2(0) = c_2,\ \ y_3(0) = c_3 \ldots y_n(0) = c_n$
Suppose

$$D_j^{\alpha} y_n(t) = \sum_{i=1}^{m} (a_i)_n h_i(t), \quad j = 1,\ 2 \ldots, \quad n = 1,\ 2 \ldots \tag{7}$$

After integrating above system of linear differential equations with respect to t from 0 to t and using initial conditions we have

$$y_n(t) = \sum_{i=1}^{m} (a_i)_n P_{i,1}(t) + c_j, \qquad n = 1,\ 2 \ldots \tag{8}$$

Now substituting equations (7) and (8) in system of linear differential equations then we get algebraic form of system of linear differential equations. After solving these equations we can calculate haar coefficients$(a_i)_j$. Then from equation (8) we can find approximate value of $y_n(t), n = 1,\ 2 \ldots$

4 Examples and results:

Example 1: Consider the following system of linear differential equations with initial conditions

$$y_1^{'}(t) = y_3(t) - \ cost,\ y_1(0) = 1 \tag{9}$$

$$y_2'(t) = y_3(t) - e^t, \; y_2(0) = 0 \tag{10}$$

$$y_3'(t) = y_1(t) - y_2(t), \; y_3(0) = 2 \tag{11}$$

Exact solution of above system is $y_1(t) = e^t, \; y_2(t) = sint, \; y_3(t) = e^t + cost$
Suppose

$$y_1'(t) = \sum_{i=1}^{m} (a_i)_1 h_i(t) = a_1^T H \tag{12}$$

$$y_2'(t) = \sum_{i=1}^{m} (a_i)_2 h_i(t) = a_2^T H \tag{13}$$

$$y_3'(t) = \sum_{i=1}^{m} (a_i)_1 h_i(t) = a_3^T H \tag{14}$$

Now integrating equation (12) – (14) with respect to t from 0 to t we get

$$y_1(t) = \sum_{i=1}^{m} (a_i)_1 P_{i,1}(t) + 1 = a_1^T P_1 + 1 \tag{15}$$

$$y_2(t) = \sum_{i=1}^{m} (a_i)_2 P_{i,1}(t) = a_2^T P_1 \tag{16}$$

$$y_3(t) = \sum_{i=1}^{m} (a_i)_3 P_{i,1}(t) + 2 = a_3^T P_1 + 2 \tag{17}$$

Now substitute values from equations (12) – (17) in equations (9) – (11) then we get

$$a_1^T H - a_3^T P_1 - 2 + cost = 0 \tag{18}$$

$$a_2^T H - a_3^T P_1 - 2 + e^t = 0 \tag{19}$$

$$a_3^T H - a_1^T P_1 - 1 + a_2^T P_1 = 0 \tag{20}$$

After solving equations (18) – (19) we have

$$a_1^T = a_3^T P_1 H^{-1} + F H^{-1} \tag{21}$$

$$a_2^T = a_3^T P_1 H^{-1} + G H^{-1} \tag{22}$$

Now from equations (20) – (22)

$$a_3^T = F H^{-1} P_1 H^{-1} - G H^{-1} P_1 H^{-1} + H^{-1} \tag{23}$$

Where F and G are discrete values of $2 - cost$ and $2 - e^t$ respectively of order $1 \times m$.
With the help of equations (21) - (23) we can find value of a_2^T, a_1^T and a_3^T. Then from equations (15)- (17) we get approximate value of $y_1(t), \; y_2(t)$ and $y_3(t)$. Now absolute error in between our method and Adomain decomposition method is shown in following table 1 and figure 1 and figure 2.

Table - 1

t/32	$y_{1(haar)} - y_{1(adom.)}$	$y_{1(haar)} - y_{1(adom.)}$	$y_{1(haar)} - y_{1(adom.)}$
1	0.000499396180104	0.000009349337806	0.000026310235501
3	0.000524451062944	0.000026257110923	0.000082348087596
5	0.000554571628228	0.000039377103342	0.000143711097876
7	0.000590185803811	0.000048292660451	0.000210559676086
9	0.000631737410419	0.000052577817734	0.000283056049077
11	0.000679686677002	0.000051797464946	0.000361365126040
13	0.000734510837021	0.000045507449291	0.000445655449932
15	0.000796704812708	0.000033254610995	0.000536100237600
17	0.000866781994559	0.000014576744485	0.000632878511192
19	0.000945275123650	0.000010997521744	0.000736176323482
21	0.001032737284564	0.000043948933628	0.000846188079866
23	0.001129743017108	0.000084767915569	0.000963117959848
25	0.001236889555230	0.000133954977384	0.001087181441002
27	0.001354798201923	0.000192021231612	0.001218606928532
29	0.001484115849219	0.000259489028844	0.001357637493701
31	0.001625516652769	0.000336892718862	0.001504532724648

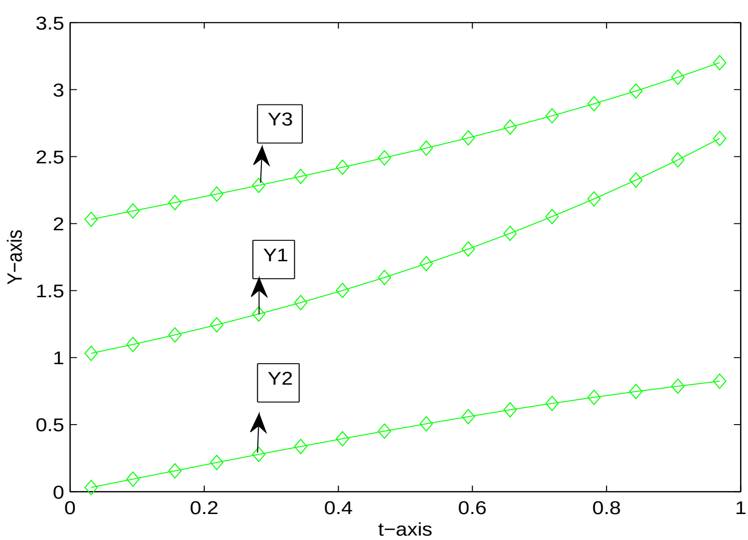

Figure 1: Approximate solution of system of linear differential equations.

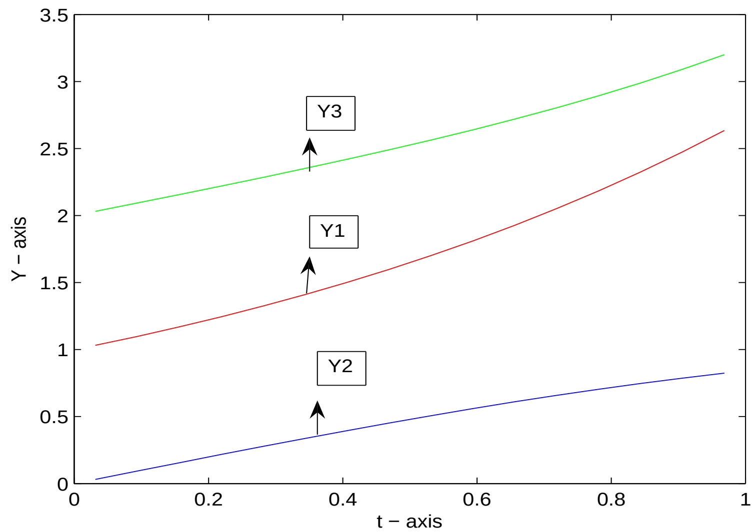

Figure 2: Exact solution of system of linear differential equations.

Example 2: (Biomass Transfer) Consider a European forest having one or two variables of trees. We select some of the oldest trees, these are expected to die off in the next few years, then follow the cycle of living trees into dead trees. The dead trees eventually decay and fall from seasonal and biological events. Finally, the fallen trees become humus. Let variables $y_1,\ y_2,\ y_3$ and t be defined by
$y_1\,(t) =$ Biomass decayed into humus,
$y_2\,(t) =$ Biomass of dead trees,
$y_3\,(t) =$ Biomass of living trees,
$t =$ time in decades
A typical biological modal is

$$y_1^{'}\,(t) = \ y_1\,(t) + 3y_2\,(t)\,, \tag{24}$$

$$y_2^{'}\,(t) = \ -3y_2\,(t) + \ 5y_3\,(t)\,, \tag{25}$$

$$y_3^{'}\,(t) = \ -\ 5y_3\,(t)\,, \tag{26}$$

Suppose there are no dead tree and no humus at $t = 0$, with initially Z_0 units of living tree biomass. These assumptions imply initial conditions

$$y_1\,(0) = 0,\ \ y_2\,(0) = 0,\ \ y_3\,(0) = Z_0$$

Exact solution of above system is $y_1\,(t) = \ \frac{15}{8}\ Z_0\left(e^{-5t}\ - 2e^{-3t} + \ e^{-t}\right),$

$$y_2\,(t) = \ \frac{5}{2}Z_0\left(-e^{-5t} + e^{-3t}\ \right),\ y_3\,(t) = Z_0\ e^{-5t}.$$

The live tree biomass $y_3\,(t) = Z_0\ e^{-5t}$ decreases according to a Mathusan decay law from its initial size Z_0. It decays to 60% of its original biomass in one year.
In our numerical solution we consider $Z_0 = 1$, $0 = t = 1$ and $m = 16$.

Suppose

$$y_1'(t) = \sum_{i=1}^{m} (a_i)_1 h_i(t) = a_1^T H \tag{27}$$

$$y_2'(t) = \sum_{i=1}^{m} (a_i)_2 h_i(t) = a_2^T H \tag{28}$$

$$y_3'(t) = \sum_{i=1}^{m} (a_i)_3 h_i(t) = a_3^T H \tag{29}$$

Now integrating equation (27) -(29) with respect to t from 0 to t we get

$$y_1(t) = \sum_{i=1}^{m} (a_i)_1 P_{i,1}(t) = a_1^T P_1 \tag{30}$$

$$y_2(t) = \sum_{i=1}^{m} (a_i)_2 P_{i,1}(t) = a_2^T P_1 \tag{31}$$

$$y_3(t) = \sum_{i=1}^{m} (a_i)_3 P_{i,1}(t) + 1 = a_3^T P_1 + 1 \tag{32}$$

After solving above equations we get

$$a_1^T = \frac{3a_2^T P_1}{(H + P_1)} \tag{33}$$

$$a_2^T = \frac{5a_3^T P_1}{(H + 3P_1)} + \frac{5I}{(H + 3P_1)} \tag{34}$$

$$a_3^T = \frac{-5I}{(H + 5P_1)} \tag{35}$$

Where I $= [1\ 1\ 1 \ldots 1]_{1\times m}$.
Now from equations (24) - (35) we can find approximate value of $y_1(t)$, $y_2(t)$ and $y_3(t)$. Now absolute error in between our method and Adomain decomposition method is shown in following table 2 and figure3 and figure 4.

Table - 2

t/32	$y_{1(haar)} - y_{1(adom.)}$	$y_{1(haar)} - y_{1(adom.)}$	$y_{1(haar)} - y_{1(adom.)}$
1	0.00456106306258	0.01436046162941	0.00951953755744
3	0.00061472371808	0.00655544016708	0.00533359448599
5	0.00134235346321	0.00202698824583	0.00271191688908
7	0.00211257271492	0.00043315348400	0.00111553880007
9	0.00221029389604	0.00162018015065	0.00018234471623
11	0.00195833977911	0.00205076721369	0.00032909711857
13	0.00155356695974	0.00205520792954	0.00057827504933
15	0.00111112464071	0.00184020073194	0.00066936063842
17	0.00069409439591	0.00153139236782	0.00066943661019
19	0.00033315172069	0.00120201139138	0.00062091870429
21	0.00003945633719	0.00089198660039	0.00054997672776
23	0.00018702110846	0.00062058968344	0.00047220882049
25	0.00035222293815	0.00039469667217	0.00039643767143
27	0.00046474725396	0.00021410600848	0.00032722871503
29	0.00053391041603	0.00007489570004	0.00026654335581
31	0.00056865859265	0.00002851263694	0.00021481048135

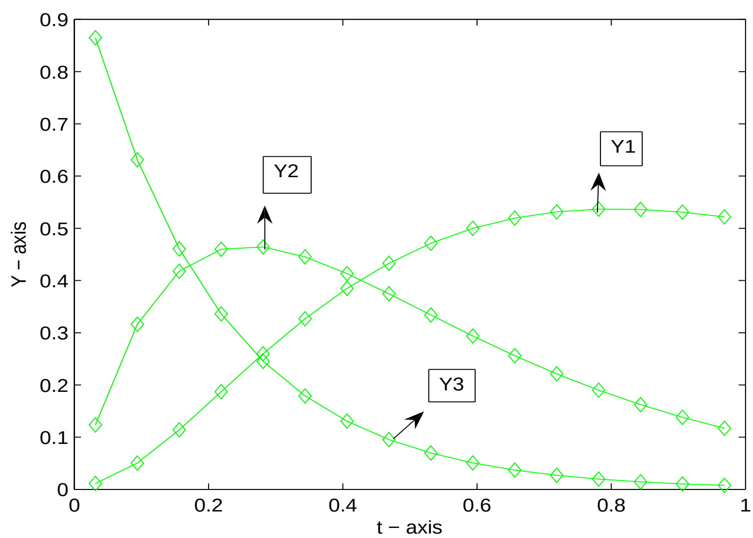

Figure 3: Approximate solution of system of linear differential equations.

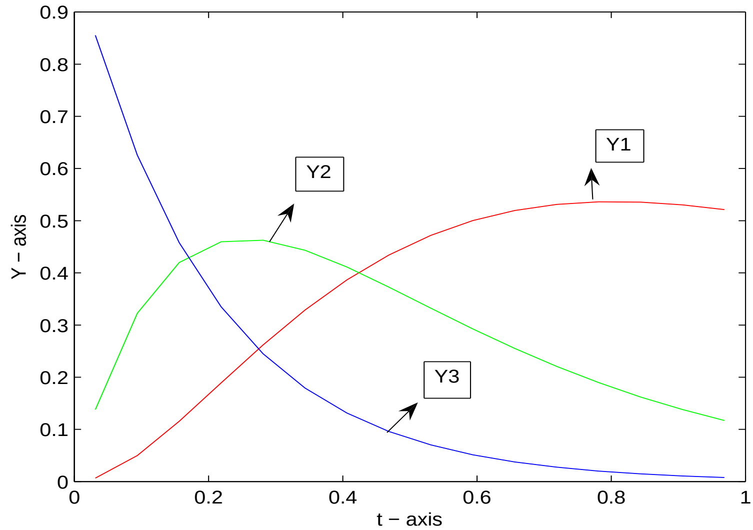

Figure 4: Exact solution of system of linear differential equations.

5 Conclusion:

The main goal of this paper is to demonstrate that the Haar wavelet operational method is a powerful tool for solving system of linear differential equations. Approximate solution of the system of linear differential equations, obtain by Matlab, are compared with exact solution. This calculation shows the accuracy of the Haar wavelet solution. Applications of this method are very simple, and also it gives the implicit form of the approximate solutions of the problems. Hence, the present method is a very reliable, simple, fast, minimal computation costs and flexible.

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