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On the behavior of certain turing system

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Abstract: *We extend the generalized maximum principle of Lou and Ni [1] of elliptic equations to parabolic equations. By this result, we show that the solution of a Turing system has a global attractor provide the diffusion coefficient $D \neq 0$ otherwise the solution blow-up in finite time.*

Keywords: *Maximum principle, Existence of solutions, Global attractor, Finite time blow-up.*

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1 Introduction

The Brusselator equation was first introduced by Prigogine and Lefever [2] and it attract many research attention due to its complicated dynamic behavior. It is a model of autocatalytic of chemical (or biological chemical [3, 4, 5, 6]) reaction. In particular, it describes an activation-depletion mechanism [7] of a reaction moreover, it is a Turing system as well. The signature of a Turing system is Hopf's bifurcation which is equivalent to the changing of the stability of the dynamics. The mathematical model of it is as follows:

$$\left\{ \begin{array}{lcl} u_t &=& D_u \Delta u + u^2 v - (\beta+1)u + \alpha \\ v_t &=& D_v \Delta v - u^2 v - \beta u, \end{array} \right. \qquad x \in \Omega, \tag{1}$$

with Neumann boundary condition

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \tag{2}$$

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and initial data

$$u(0,\cdot) = u_0, \quad v(0,\cdot) = v_0. \tag{3}$$

The history about the research of this model is long. Recently, Ghergu [8] show the existence, non existence and regularity of the steady states of (1) by generalized maximum principle of Lou and Ni [1]. Many other then focus on the dynamic model without diffusion term. In this article, we will present a complete study of the asymptotic behavior of this model related to the diffusion coefficient D instead of the parameter α and β.

To clarify the changing of the asymptotic behavior due to the varying of parameters of the system we rescale the variables by $s = kt$, $y = kx$ and $u' = \frac{u}{k}$, $v' = \frac{v}{k}$. Without ambiguity, we still use notation t, x, and u, v instead of s, y, and u', v' and we let $D = d/k$ then system (1) becomes;

$$\begin{cases} u_t &= D\Delta u + \frac{u^2 v}{k^2} - (\beta+1)u + \alpha \\ v_t &= \Delta v - \frac{u^2 v}{k^2} - \beta u. \end{cases} \quad x \in \Omega. \tag{4}$$

2 Global attractor

Throughout this article, we will assume $\mu(\Omega) = 1$, where $\mu(\cdot)$ is the measure of R^2.

For the purpose of self complementary, we give a brief proof of the extension the generalized maximum principle of parabolic equation (cf. Lou and Ni [1]). To this end, we denote $E_T = \{(x,t) | x \in \Omega, \text{ and } t \in (0,T]\}$.

Proposition 1. *Let $g \in C(\bar{\Omega} \times (0,T) \times R)$ and $w(x,0) \geq 0$.*

(i) If $w \in C^{2,1}(\Omega \times (0,T)) \cap C^{1,0}(\bar{\Omega} \times [0,T])$ satisfies

$$-w_t + \Delta w + g(x,t,w) \geq 0, \quad \frac{\partial w}{\partial n}|_{\partial\Omega} \leq 0 \tag{5}$$

and $w(x_0,t_0) = \max_{(x,t)\in\bar{\Omega}\times[0,T]} w(x,t)$; then, $g(x_0,t_0,w(x_0,t_0)) \geq 0$.

(ii) If $w \in C^{2,1}(\Omega \times (0,T)) \cap C^{1,0}(\bar{\Omega} \times [0,T])$ satisfies

$$-w_t + \Delta w + g(x,t,w) \leq 0, \quad \frac{\partial w}{\partial n}|_{\partial\Omega} \geq 0, \tag{6}$$

and $w(x_0,t_0) = \min_{(x,t)\in\bar{\Omega}\times[0,T]} w(x,t)$; then, $g(x_0,t_0,w(x_0,t_0)) \leq 0$.

Proof. We will only prove (i), since (ii) can be derived similarly. Let (x_0,t_0) be an interior point of domain $\Omega \times (0,T)$ such that $w(x_0,t_0) = \max_{\xi\in\Omega\times(0,T)} w(\xi)$. From (6) $g(x_0,t_0,w(x_0,t_0)) \geq w_t(x_0,t_0) - \Delta w(x_0,t_0) \geq 0$.

To prove the case of the boundary point $(x_0,t_0) \in \partial\bar{\Omega} \times [0,T]$, we argue by contradiction (cf. [9]) and we assume that $g(x_0,t_0,w(x_0,t_0)) < 0$ where $w(x_0,t_0) = \max_{\xi\in\partial\bar{\Omega}\times[0,T]} w(\xi)$. By (5), $g(x_0,t_0,w(x_0,t_0)) + \Delta w(x_0,t_0,w(x_0,t_0)) \geq w_t$, we have $w_t < 0$, thus, w is larger at eariler time. We derive a contradiction. Thus (i) is proved. □

With the above result, we obtain the existence of global attractor for D closed to 1.

Theorem 1. *If u_0, v_0 are non-negative and the diffusion coefficient $|D-1| \leq \epsilon$, for some $\epsilon > 0$, then (4) has a unique positive classical solution with global attractor satisfying:*

$$\begin{aligned} \frac{\alpha}{1+\beta} \leq \; & u \leq \frac{\alpha^2 + k^2\beta(\beta+1)}{\alpha}, \\ \frac{\alpha\beta}{\alpha^2 + k^2\beta(\beta+1)} \leq \; & v \leq \frac{k^2\beta(\beta+1)}{\alpha}. \end{aligned}$$

Proof. We begin with $D = 1$, and let $u(p_n) = \min_{\xi\in\bar{\Omega}\times(0,T)} u(\xi)$; then, by (ii) of lemma 1, we have

$$\begin{array}{rcl} 0 & \geq & \alpha - (1+\beta)u(p_n) + \frac{u^2(p_n)v(p_n)}{k^2} \\ & \geq & \alpha - (1+\beta)u(p_n) \end{array}$$

thus

$$u(p_n) \geq \frac{\alpha}{1+\beta}. \tag{7}$$

Let $v(q_m) = \max_{\xi\in\Omega\times(0,T)} v(\xi)$; then, by (i) of lemma 1 and (7), we have

$$v(q_m) \leq \frac{k^2\beta(\beta+1)}{\alpha}. \tag{8}$$

We define $w = u + v$; then,

$$0 = -w_t + \Delta w + \alpha - u. \tag{9}$$

Let $w(r_m) = \max_{\xi\in\Omega\times(0,T)} w(\xi)$; then, by (i) of lemma 1, $u(r_m) \leq \alpha$. Hence,

$$\begin{array}{rcl} u(x,t) \leq w(x,t) & \leq & w(r_m) \\ & = & u(r_m) + v(r_m) \\ & \leq & \alpha + \frac{k^2\beta(\beta+1)}{\alpha}. \end{array} \tag{10}$$

Let $v(q_n) = \min_{\xi\in\Omega\times(0,T)} v(\xi)$; then, by (ii) of lemma 1, we have $\beta u(q_n) \leq u^2(q_n)v(q_n)$. Hence,

$$v(x,t) \geq v(q_n) \geq \frac{\beta}{u(q_n)} \geq \frac{\alpha\beta}{\alpha^2 + k^2\beta(\beta+1)}. \tag{11}$$

To attain the global solution, we let

$$L_D = \begin{pmatrix} D\Delta & 0 \\ 0 & \Delta \end{pmatrix}.$$

We consider Banach space $X = C(\Omega)\times C(\Omega)$ and let $U = (u, v)$, and we denote $F(\alpha, \beta, u, v) = F(\alpha, \beta, U)$; then, (4) may rewrite as follows:

$$\begin{array}{rcl} U_t & = & L_D U + F(\alpha, \beta, U) \\ U_0 & = & (u_0, v_0) \in X, \end{array} \tag{12}$$

Since Δ generates a contraction semi-group on $C(\Omega)$, so does the direct sum L_D. Thus the existence of unique local solution of equation (12) is established. By standard results of parabolic equation [10], the solution of (4) is classical. □

We could extend Theorem 2 further to $D \neq 0$, or equivalently $k \to \infty$. However, we prefer to discuss after the critical case that $D = 0$ at the end of this article.

3 Blow-up Solutions

In this section, we will study the asymptotic behavior of the solution that affected by the diffusion coefficient D. We begin with the case when the diffusion coefficient $D = 0$ and equation (4) then reduced to:

$$\left\{ \begin{array}{rcl} u_t & = & u^2 v - (\beta+1)u + \alpha \\ v_t & = & \Delta v - u^2 v + \beta u, \end{array} \right. \qquad x \in \Omega, \tag{13}$$

while the boundary condition and initial data remain the same.

Lemma 1. *The solution u is alway positive and blow-up in finite time, furthermore, there exists a positive constant M_v such that $0 \le v < M_v$.*

Proof. Let $X = C(\Omega) \times C(\Omega)$ and

$$L = \begin{pmatrix} -\beta - 1 & 0 \\ 0 & \Delta \end{pmatrix} \equiv \begin{pmatrix} L_1, & 0 \\ 0 & L_2 \end{pmatrix}.$$

Since L_1 and L_2 both generate contractive semi-groups on $C(\Omega)$, so does L. Thus the existence and uniqueness of local solution of equation (13) is established.

To show that the solution u of the first equation of (13) is unbounded, we will argue by contradiction. We assume that u is bounded. In this case, we claim that v is bounded away from zero, that is $v > m_v \ge 0$. As in section 1, we let $q_v \in \Omega \times (0, T)$ and $v(q_v) = \min_{(x,t)\in E_T} v(x,t)$ then by the generalized maximum principle Proposition 1,

$$0 \ge -u^2(q_v)v(q_v) + \beta u(q_v) = u(q_v)(-u(q_v)v(q_v) + \beta). \tag{14}$$

Since u is bounded, we have $v(q_v) \ge \frac{\beta}{u(q_v)} \ge \frac{\beta}{u(Q_u)} > 0$ where $u(Q_u) = \max_{(x,t)\in E_T} u(x,t)$.

Furthermore, we let $\bar{v} = \frac{\beta}{u(Q_u)}$ and

$$I_u = \int_\Omega u dx,$$

then

$$\frac{dI_u}{dt} = \int_\Omega u^2 v - (\beta+1)u + \alpha dx \ge \int_\Omega u^2\bar{v} - (\beta+1)u dx.$$

Let $y(t)$ be the solution of

$$y' = \bar{v}y^2 - (\beta+1)y,$$

then y blow-up in finite time and so does u that contradicts to our assumption that u is bounded.

By (14), $v \ge v_m$ is bounded from below and $v_m = 0$ if $\sup_t u = \infty$. To prove that v is bounded above we let $v(Q_v) = \max_{(x,t)\in E_T} v(x,t)$ and $u(q_u) = \min_{(x,t)\in E_T} u(x,t)$ then again by Proposition 1,

$$u^2(Q_v)v(Q_v) \le \beta u(Q_v),$$

thus

$$v(Q_v) \le \frac{\beta}{u(q_u)}.$$

Thus v is bounded from above if $u(q_u) > 0$. To prove $u(q_u) > 0$, we apply the quadratic formula to the right hand side of u and by v is non-negative we have,

$$r_\pm = \frac{(\beta+1) \pm \sqrt{(\beta+1)^2 - 4v\alpha}}{2v} > 0. \tag{15}$$

The radius of the global absorbing set of v may obtain directly from the Lyapunov function

$$I_v = \frac{1}{2}\int_\Omega v^2 dx.$$

Thus we omit the proof. □

From lemma 3, we would expect that solution u blow-up in finite time if D is small due to the nonlinearity reaction term but on the contrary the blow up behavior occurs only when $D = 0$.

Lemma 2. *If the diffusion coefficient $D \neq 0$ then the solution has a global attractor.*

Proof. Let $w = u + v$ then

$$w_t = \Delta w - (1 - D)\Delta u - u + \alpha. \tag{16}$$

Integrating (16) over Ω, we obtain

$$\frac{\partial}{\partial t}\int_\Omega w dx = -\int_\Omega w dx + \int_\Omega v dx + \alpha,$$

By lemma 7, v is bounded thus $\int_\Omega v dx + \alpha \leq k$ for some k. Let $y(t)$ be the solution of

$$y_t = -y + k,$$

then $\int_\Omega w dx \leq y$ and w has a global attractor. Since v is bounded, u also has a global attractor. □

4 Conclusion

Examining the simulation of the solution of system (4) we find many interesting phenomenons. For example, the mesa type of solution was done by Kolokolnikov et al. [4] using perturbation method. The localization is another interesting phenomenon of the solution of heat equation. In fact, the main reason that we study the blow-up behavior of the solution is to resolve such a phenomenon. However, our result answers only part of the question. The main difficulty is that the solution of occurrence of the blow-up behavior is the ordinary differential equation of system (4) but not the parabolic equation. An ordinary differential equation does not exhibit the behavior of the solution concerning spatial variable. In fact, the localization of system (4) remains open.

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