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Application of Double Natural Transform to One-Dimensional Heat Equations

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Abstract
In this paper, we exhibit the application of double Natural transform method to solve one-dimensional heat equations $\frac{\partial U(x,t)}{\partial t} = \frac{\partial^2 U(x,t)}{\partial x^2} + f(x,t)$ under the given conditions. The Double Natural Transform method is more general method and so provide wider domain of the application and it can be observed by the work of Eshag [6], that, the obtained results are comparable with the solutions those were obtained in the literature with the help of double Laplace and double Sumudu transform methods. The results calculated numerically by using suitable values of parameters and we interpret these results graphically.

Keywords
double laplace transform, double natural transform, double sumudu transform, natural transform, one dimensional heat equation.

2010 Mathematical Subject classification: 35C05, 35C09, 44A15, 44A30

1. Introduction

Partial differential equations and their applications are very useful in various branches of engineering and basic sciences. Many researchers have done their work on the solutions of various types of Partial differential equations with initial as well as boundary conditions by using different methods. The list of methods may include numerical methods like finite difference method, finite element method, finite volume method etc., analytical methods like separation of variables method, method of characteristics etc., calculus of variations method, charpit method, integral transform methods like Laplace Transform, Fourier Transform, Mellin Transform, Hankel Transform etc.

Heat transfer is the process of transferring heat energy from one point to another. In this process, heat flows from the point of higher temperature to the point of lower temperature. The governing equation of this heat transfer is known as heat equation, this equation is a parabolic partial differential equation that describes the distribution of heat (or variation in temperature) in a given region over time.

In this paper, we have solved one dimensional heat equations by using Double Natural Transform (DNT) method and exhibit those numerically and graphically by using Origin Lab. These results are comparable with the results those have been obtained earlier in the literature by double Laplace and double Sumudu transform method [5, 6, 9].

The Natural transform was developed and studied by Khan and Khan [8] and some of its properties and applications were investigated by Al-Omari [1], Belgacem and Silambarasan [2, 3, 4] and Mana and Omranb [10]. The Natural transform usually deals with continuous and continuously differentiable functions, if the function is not differentiable then the Natural transform fails to apply, similarly as Laplace transform [11] and Sumudu transform [13]. The main purpose of this work is to generalize the definition of single Natural transform to double Natural transform and achieve its main properties, in order to solve one dimensional heat equation.

The Natural transform [8] of a function $f(t)$ is given by:

$$N[f(t)] = R(s,u) = \int_0^\infty e^{-st} f(ut)dt = \frac{1}{u}\int_0^\infty e^{-s\frac{t}{u}} f(t)dt \quad \ldots (1.1)$$

Remark 1.1: if we take $u = 1$ and $s = 1$, then the Natural transform reduces in Laplace Transform [11] and Sumudu transform [13] respectively.

The Double Natural Transform (DNT) [10] of a function $f(x,y)$; $x,y \in R_+$ is defined as

$$N_+^2[f(x,y)] = R_+^2[(s,p);(u,v)] = \int_0^\infty \int_0^\infty e^{-(sx+py)} f(ux,vy)dxdy == \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\left(\frac{sx}{u}+\frac{py}{v}\right)} f(x,y)dxdy \quad \ldots (1.2)$$

Remark 1.2: If we take $u = v = 1$ in (1.2), we get Double Laplace Transform [7] as

$$L^2[f(x,y)] = F^2(s,p) = \int_0^\infty \int_0^\infty e^{-(sx+py)} f(x,y)dxdy \, ; x,y \in R_+ \quad \ldots (1.3)$$

Remark 1.3: If we take $s = p = 1$ in (1.2), we get Double Sumudu transform [12] as

$$S^2\{f(x,y)\} = G^2(u,v) = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\left(\frac{x}{u}+\frac{y}{v}\right)} f(x,y)dxdy \, ; x,y \in R_+ \quad \ldots (1.4)$$

2. Main Results

In this section, we assume that the inverse double Natural transform exist for the functions those are used in the problems, mentioned in the following part of this section. Then we have applied the double Natural transform (DNT) to find the solution of the one-dimensional heat equation of type $\frac{\partial U(x,t)}{\partial t} = \frac{\partial^2 U(x,t)}{\partial x^2} + f(x,t)$ with initial and boundary conditions. Such problems are very common in Mechanical engineering and Thermal engineering, out of those some problems are taken here for the said purpose.

Problem 2.1 Let us consider the heat equation (one can refer [6]) given by

$$\frac{\partial U(x,t)}{\partial t} = \frac{\partial^2 U(x,t)}{\partial x^2}, \; t > 0 \quad \ldots (2.1)$$

with conditions

$$U(0,t) = 0 \; , \quad U(x,0) = \sin x \quad , \frac{\partial U(0,t)}{\partial x} = e^{-t} \quad \ldots (2.2)$$

Solution: By taking the DNT of equation (2.1) from both the sides, we get

$$\frac{p}{v} N_+^2\{U(x,t)\} - \frac{N_+^2\{U(x,0)\}}{v} = \frac{s^2}{u^2} N_+^2\{U(x,t)\} - \frac{s}{u^2} N_+^2\{U(0,t)\} - \frac{1}{u} N_+^2\left\{\frac{\partial U(0,t)}{\partial x}\right\} \quad \ldots (2.3)$$

by taking Natural transform on initial conditions (2.2) gives

$$N\{U(0,t)\} = 0 \quad , N\{U(x,0)\} = \frac{u}{s^2+u^2} \quad , N\left\{\frac{\partial U(0,t)}{\partial x}\right\} = \frac{1}{p+v}$$

…
(2.4)

from (2.4) and (2.3) we get

$$\left[\frac{p}{v}-\frac{s^2}{u^2}\right][N_+^2\{U(x,t)\}]=\frac{u}{v(s^2+u^2)}-\frac{1}{u(p+v)}$$

$$[N_+^2\{U(x,t)\}]=\frac{u}{(p+v)(s^2+u^2)}$$

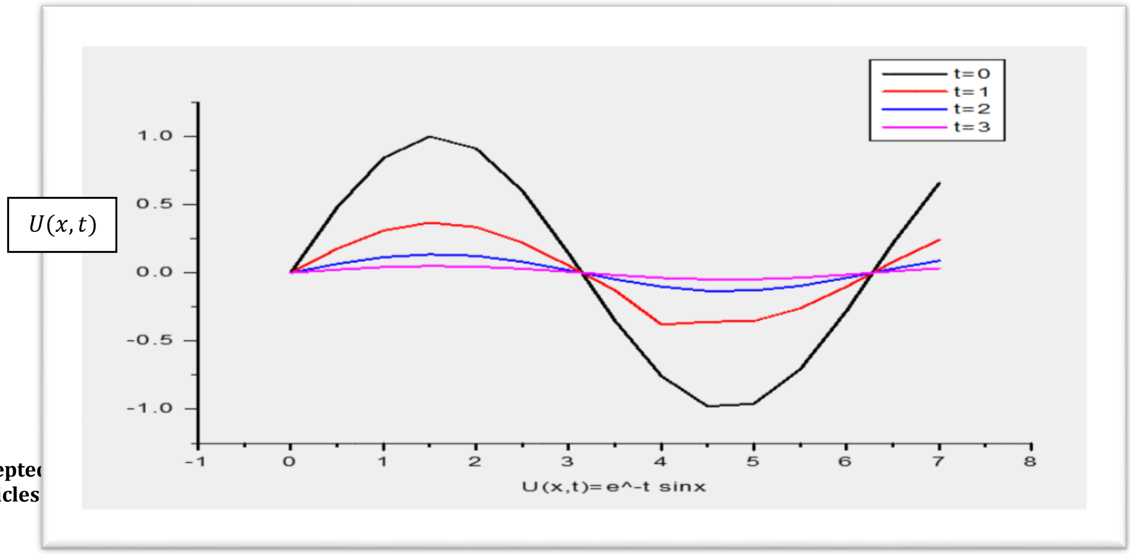
… (2.5)
Now by taking inverse double natural transform we get the solution of (2.1) as

$$U(x,t)=e^{-t}\sin x$$

… (2.6)
The graphs of the solution of the heat equation (2.1) are presented as Fig. 2.1 and Fig. 2.2, in which we fix t and x one by one and get the numerical results which are shown here in Table 2.1 and Table 2.2. The range of x and t are $0 \le x \le 7.5$ and $0 \le t \le 3$ respectively.

Table 2.1: Values of $U(x,t)$ for fixed t

Fix $t=0$		Fix $t=1$		Fix $t=2$		Fix $t=3$	
(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$
(0,0)	0	(0,1)	0	(0,2)	0	(0,3)	0
(0.5,0)	.4794	(0.5,1)	.1764	(0.5,2)	.0649	(0.5,3)	.0239
(1,0)	.8415	(1,1)	.3096	(1,2)	.1139	(1,3)	.0419
(1.5,0)	.9975	(1.5,1)	.3670	(1.5,2)	.1350	(1.5,3)	.0497
(2, 0)	.9093	(2, 1)	.3345	(2, 2)	.12306	(2, 3)	.0453
(2.5, 0)	.5985	(2.5, 1)	.2202	(2.5,2)	.0810	(2.5,3)	.0298
(3, 0)	.1411	(3, 1)	.05192	(3, 2)	.0191	(3, 3)	.0070
(3.5, 0)	-.3508	(3.5, 1)	-.1290	(3.5,2)	-.0475	(3.5,3)	-.0175
(4, 0)	-.7568	(4, 1)	-.3784	(4, 2)	-.1024	(4, 3)	-.0377
(4.5, 0)	-.9775	(4.5, 1)	-.3596	(4.5,2)	-.1323	(4.5,3)	-.0487
(5,0)	-.9589	(5,1)	-.3528	(5,2)	-.1298	(5,3)	-.0478
(5.5, 0)	-.7055	(5.5, 1)	-.2596	(5.5,2)	-.0955	(5.5,3)	-.0351
(6, 0)	-.2794	(6, 1)	-.1028	(6, 2)	-.0378	(6, 3)	-.0139
(6.5, 0)	.2151	(6.5, 1)	.0791	(6.5, 2)	.0291	(6.5,3)	.0107
(7, 0)	.6570	(7, 1)	.2417	(7, 2)	.08891	(7,3)	.0327
(7.5,0)	.938	(7.5,1)	.3451	(7.5,2)	.1269	(7.5,3)	.0467



x

Figure 2.1: Graphical Interpretation of $U(x,t)$ for fixed t

Discussion: By observing the numerical values in Table 2.1 and the figure 2.1, that, for fix values of t, the heat flow moves like sine wave and as the time passes, the amplitude of the wave form flow reduced gradually and moving towards cool down situation.

Table 2.2 : Values of $U(x,t)$ for fixed x

Fix $x=0$		Fix $x=1.5$		Fix $x=3$		Fix $x=4.5$		Fix $x=6$		Fix $x=7.5$	
(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$
(0,0)	0	(1.5,0)	.09975	(3,0)	.1411	(4.5,0)	-.9775	(6,0)	-.2794	(7.5,0)	.9380
(0,1)	0	(1.5,1)	.3670	(3,1)	.0519	(4.5,1)	-.3596	(6,1)	-.1028	(7.5,1)	.3451
(0,2)	0	(1.5,2)	.13500	(3,2)	.0191	(4.5,2)	-.1323	(6,2)	-.03781	(7.5,2)	.1269
(0,3)	0	(1.5,3)	.04966	(3,3)	.0070	(4.5,3)	-.0487	(6,3)	-.0139	(7.5,3)	.0467
(0,4)	0	(1.5,4)	.01827	(3,4)	.0026	(4.5,4)	-.01790	(6,4)	-.0051	(7.5,4)	.0171
(0,5)	0	(1.5,5)	.00672	(3,5)	.00095	(4.5,5)	-.0066	(6,5)	-.0019	(7.5,5)	.0063
(0,6)	0	(1.5,6)	.0025	(3,6)	.00035	(4.5,6)	-.0024	(6,6)	-.0007	(7.5,6)	.0023

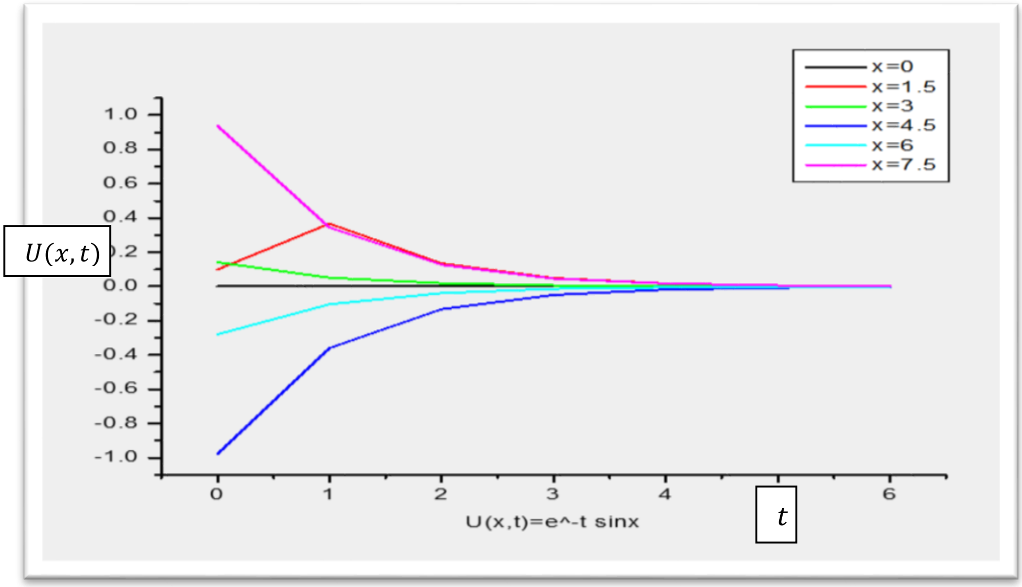


Figure 2.2: Graphical Interpretation of $U(x,t)$ for fixed x

Discussion: By observing the numerical values in Table 2.2 and the figure 2.2, that, for fix values of x, the heat flow moves like the exponential wave and moving towards cool down situation as time passes.

Problem 2.2 Let us consider the heat equation

$$\frac{\partial U(x,t)}{\partial t} = \frac{\partial^2 U(x,t)}{\partial x^2} + \sin x, \quad t > 0$$

… (2.7)

With conditions

$$U(0,t) = e^{-t}\ ,\quad U(x,0) = \cos x \ \ and \ \ \frac{\partial U(0,t)}{\partial x} = 1 - e^{-t}$$

… (2.8)

Solution: Applying DNT on the equation (2.7) in both the sides, we get

$$\frac{p}{v}N_+^2\{U(x,t)\} - \frac{1}{v}N_+^2\{U(x,0)\} = \frac{s^2}{u^2}N_+^2\{U(x,t)\} - \frac{s}{u^2}\ \ N_+^2\{U(0,t)\}\ \ - \frac{1}{u}\ N_+^2\left\{\frac{\partial U(0,t)}{\partial x}\right\} + \frac{u}{p(s^2+u^2)}$$

… (2.9)

Also by taking Natural transform on initial conditions (2.8) gives

$$N[U(0,t)] = \frac{1}{p+v}\quad ,N[U(x,0)] = \frac{s}{s^2+u^2}\quad ,N[\frac{\partial u(0,t)}{\partial x}] = \frac{1}{p} - \frac{1}{p+v}$$

… (2.10)

Substitute all these values in (2.9) we get

$$\left[\frac{p}{v} - \frac{s^2}{u^2}\right][N_+^2\{U(x,t)\}] = \frac{s}{v(s^2+u^2)} - \frac{s}{u^2(p+v)} - \frac{1}{u}\left(\frac{1}{p} - \frac{1}{p+v}\right) + \frac{u}{p(s^2+u^2)}$$

$$[N_+^2\{U(x,t)\}] = \frac{s}{(p+v)(s^2+u^2)} + \frac{u}{p(s^2+u^2)}\cdot\frac{v}{(p+v)}$$

…

(2.11)

Now by taking inverse double natural transform we get the solution of (2.7) as

$$U(x,t) = e^{-t}\cos x + \sin x\,(1 - e^{-t})$$

…(2.12)

The graphs of the solution of the heat equation (2.7) are presented as Fig. 2.3 and Fig. 2.4, in which we fix t and x one by one respectively and get the numerical results which are shown here in Table 2.3 and Table 2.4. The range of x and t are $0 \le x \le 7.5$ and $0 \le t \le 3$ respectively.

Table 2.3: Values of U(x,t) for fixed *t*

Fix $t = 0$		**Fix $t = 1$**		**Fix $t = 2$**		**Fix $t = 3$**	
(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$
(0,0)	1	(0,1)	.3679	(0,2)	.1353	(0,3)	.0498
(0.5,0)	.8776	(0.5,1)	.6259	(0.5,2)	.5333	(0.5,3)	.4992
(1,0)	.54030	(1,1)	.7307	(1,2)	.8685	(1,3)	.8265
(1.5,0)	.07074	(1.5,1)	.6566	(1.5,2)	.8721	(1.5,3)	.95135
(2, 0)	-.4161	(2, 1)	.4217	(2, 2)	.7299	(2, 3)	.8433
(2.5, 0)	-.8011	(2.5, 1)	.0836	(2.5,2)	.4091	(2.5,3)	.5288
(3, 0)	-.9900	(3, 1)	-.275	(3, 2)	-.012	(3, 3)	.0848
(3.5, 0)	-.9365	(3.5, 1)	-.5662	(3.5,2)	-.4300	(3.5,3)	-.37994
(4, 0)	-.6536	(4, 1)	-.7189	(4, 2)	-.7428	(4, 3)	-.7517
(4.5, 0)	-.2108	(4.5, 1)	-.6955	(4.5,2)	-.8738	(4.5,3)	-.9394
(5,0)	.2837	(5,1)	-.50180	(5,2)	-.7908	(5,3)	-.8971
(5.5, 0)	.7087	(5.5, 1)	-.1853	(5.5,2)	-.5141	(5.5,3)	-.6351

(6, 0)	.96017	(6, 1)	.1766	(6, 2)	-.1117	(6, 3)	-.2177
(6.5, 0)	.9766	(6.5, 1)	.4952	(6.5, 2)	.3182	(6.5,3)	.2530
(7, 0)	.7539	(7, 1)	.6926	(7, 2)	.67010	(7,3)	.6618
(7.5,0)	.3466	(7.5,1)	.7204	(7.5,2)	.85797	(7.5,3)	.9086

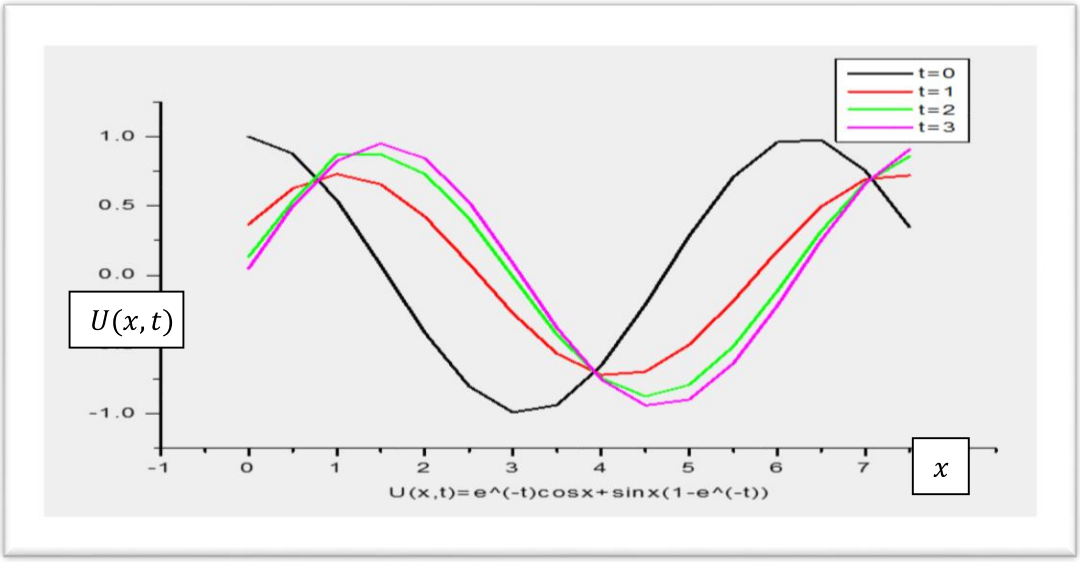


Figure 2.3: Graphical Interpretation of $U(x,t)$ for fixed t

Discussion: By observing the numerical values in Table 2.3 and the figure 2.3, that, for fix values of t, the heat flow moves like a mixture of sine and cosine wave and overlaps at a point near to $x = 1,4,7$ as the time passes.

Table 2.4: Values of $U(x,t)$ for fixed x

Fix $x = 0$		Fix $x = 1.5$		Fix $x = 3$		Fix $x = 4.5$		Fix $x = 6$		Fix $x = 7.5$	
(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$
(0,0)	1	(1.5,0)	.07074	(3,0)	-.99	(4.5,0)	-.2108	(6,0)	.9601	(7.5,0)	.3466
(0,1)	.3679	(1.5,1)	.6566	(3,1)	-.275	(4.5,1)	-.6955	(6,1)	.1766	(7.5,1)	.7204
(0,2)	.1353	(1.5,2)	.8721	(3,2)	-.012	(4.5,2)	-.8738	(6,2)	-.1117	(7.5,2)	.858
(0,3)	.0498	(1.5,3)	.95135	(3,3)	.0848	(4.5,3)	-.9394	(6,3)	-.2177	(7.5,3)	.909

(0,4)	.0183	(1.5,4)	.9805	(3,4)	.1204	(4.5,4)	-.9635	(6,4)	-.2567	(7.5,4)	.927
(0,5)	.0067	(1.5,5)	.9913	(3,5)	.1335	(4.5,5)	-.9724	(6,5)	-.2711	(7.5,5)	.934
(0,6)	.0025	(1.5,6)	.9952	(3,6)	.1383	(4.5,6)	-.9756	(6,6)	-.2763	(7.5,6)	.937

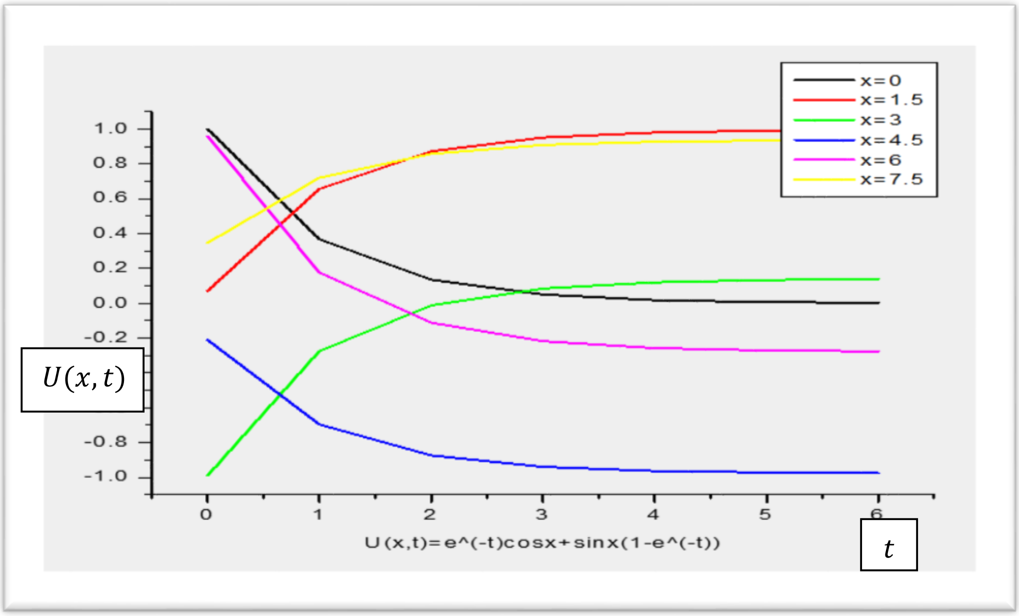


Figure 2.4: Values of $U(x,t)$ for fixed x

Discussion: By observing the numerical values in Table 2.4 and the figure 2.4, that, for fix values of x, the heat flow moves like a mixture of the exponential, sine and cosine wave and moving towards cool down situation as time passes.

Problem 2.3 Let us consider the heat equation

$$\frac{\partial U(\boldsymbol{x},\boldsymbol{t})}{\partial t} = \frac{\partial^2 U(\boldsymbol{x},\boldsymbol{t})}{\partial x^2} - 3U(\boldsymbol{x},\boldsymbol{t}) + 3, \quad t > 0 \quad \dots (2.13)$$

With conditions

$$U(0,t) = 1\ , \quad U(x,0) = 1 + sin\, x \quad , \frac{\partial U(0,t)}{\partial x} = e^{-4t} \quad \dots (2.14)$$

Solution: Applying DNT on the equation (2.13) in both the sides, we get

$$\frac{p}{v} N_+^2\{U(x,t)\} - \frac{N_+^2\{U(x,0)\}}{v} = \frac{s^2}{u^2} N_+^2\{U(x,t)\} - \frac{s}{u^2} \ N_+^2\{U(0,t)\} \ - \frac{1}{u} \ N_+^2\left\{\frac{\partial U(0,t)}{\partial x}\right\} \ - \ 3N_+^2[U(x,t)] + 3 \ N_+^2[1] \quad \dots (2.15)$$

Also, by taking Natural transform on initial conditions (2.14), we get

$$N[U(0,t)] = \frac{1}{p} \quad , N[U(x,0)] = \frac{1}{s} + \frac{u}{s^2 + u^2} \quad , N\left\{\frac{\partial u(0,t)}{\partial x}\right\} = \frac{1}{p + 4v}$$

… (2.16)

from (2.15) and (2.16), we get

$$\left[\frac{p}{v}-\frac{s^2}{u^2}+3\right][N_+^2\{U(x,t)\}]=\frac{1}{sv}+\frac{u}{v(s^2+u^2)}-\frac{1}{u^2p}-\frac{1}{u(p+4v)}+\frac{3}{sp}$$

$$[N_+^2\{U(x,t)\}]=\frac{1}{sp}+\frac{u}{(s^2+u^2)}\cdot\frac{1}{(p+4v)}$$

… (2.17)

Now by taking inverse double natural transform on equation (2.17) we get

$$U(x,t)=1+e^{-4t}\sin x$$

… (2.18)

which is the solution of given heat equation (2.13).

The graphs of the solution of the heat equation (2.13) are presented as Fig. 2.5 and Fig. 2.6, In which we fix t and x one by one respectively and get the numerical results which are shown here in Table 2.5 and Table 2.6. The range of x and t are $0 \le x \le 7.5$ and $0 \le t \le 3$ respectively.

Table 2.5: Values of $U(x,t)$ for fixed t

Fix $t = 0$		Fix $t = 1$		Fix $t = 2$		Fix $t = 3$	
(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$
(0,0)	1	(0,1)	1	(0,2)	1	(0,3)	1
(0.5,0)	1.4794	(0.5,1)	1.0088	(0.5,2)	1.00016	(0.5,3)	1
(1,0)	1.8415	(1,1)	1.0154	(1,2)	1.0003	(1,3)	1
(1.5,0)	1.9975	(1.5,1)	1.0183	(1.5,2)	1.0003	(1.5,3)	1.00001
(2, 0)	1.9093	(2, 1)	1.0167	(2, 2)	1.0003	(2, 3)	1
(2.5, 0)	1.5985	(2.5, 1)	1.011	(2.5,2)	1.0002	(2.5,3)	1
(3, 0)	1.1411	(3, 1)	1.003	(3, 2)	1.0005	(3, 3)	1
(3.5, 0)	.6492	(3.5, 1)	.9936	(3.5,2)	.9999	(3.5,3)	1
(4, 0)	.2432	(4, 1)	.9861	(4, 2)	.9997	(4, 3)	1
(4.5, 0)	.0225	(4.5, 1)	.9821	(4.5,2)	.9997	(4.5,3)	1
(5,0)	.0411	(5,1)	.9824	(5,2)	.9997	(5,3)	1
(5.5, 0)	.2945	(5.5, 1)	.9871	(5.5,2)	.9998	(5.5,3)	1
(6, 0)	.72058	(6, 1)	.9949	(6, 2)	.9999	(6, 3)	1
(6.5, 0)	1.2151	(6.5, 1)	1.0039	(6.5, 2)	1.00007	(6.5,3)	1
(7, 0)	1.6569	(7, 1)	1.0120	(7, 2)	1.00022	(7,3)	1
(7.5,0)	1.938	(7.5,1)	1.0172	(7.5,2)	1.0003	(7.5,3)	1

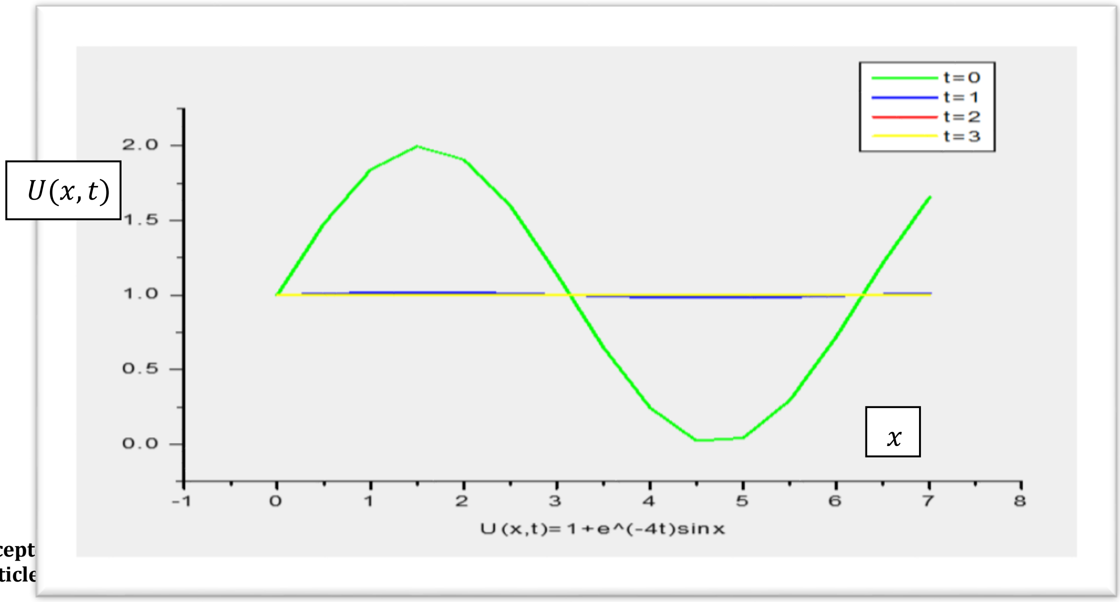


Figure 2.5: Graphical Interpretation of $U(x, t)$ for fixed t

Discussion: By observing the numerical values in Table 2.5 and the figure 2.5, that, for fix values of t, the heat flow moves like sine wave and as the time passes, the amplitude of the wave form flow reduced gradually and moving towards a constant value.

Table 2.6: Values of $U(x, t)$ for fixed x

Fix $x = 0$		Fix $x = 1.5$		Fix $x = 3$		Fix $x = 4.5$		Fix $x = 6$		Fix $x = 7.5$	
(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$	(x,t)	$U(x,t)$
(0,0)	1	(1.5,0)	1.9975	(3,0)	1.1411	(4.5,0)	.0225	(6,0)	.72058	(7.5,0)	1.938
(0,1)	1	(1.5,1)	1.0183	(3,1)	1.003	(4.5,1)	.9821	(6,1)	.9949	(7.5,1)	1.0172
(0,2)	1	(1.5,2)	1.0003	(3,2)	1.00005	(4.5,2)	.9997	(6,2)	.9999	(7.5,2)	1.0003
(0,3)	1	(1.5,3)	1.00001	(3,3)	1	(4.5,3)	.99999	(6,3)	1	(7.5,3)	1
(0,4)	1	(1.5,4)	1	(3,4)	1	(4.5,4)	.99998	(6,4)	1	(7.5,4)	1
(0,5)	1	(1.5,5)	1	(3,5)	1	(4.5,5)	1	(6,5)	1	(7.5,5)	1
(0,6)	1	(1.5,6)	1	(3,6)	1	(4.5,6)	1	(6,6)	1	(7.5,6)	1

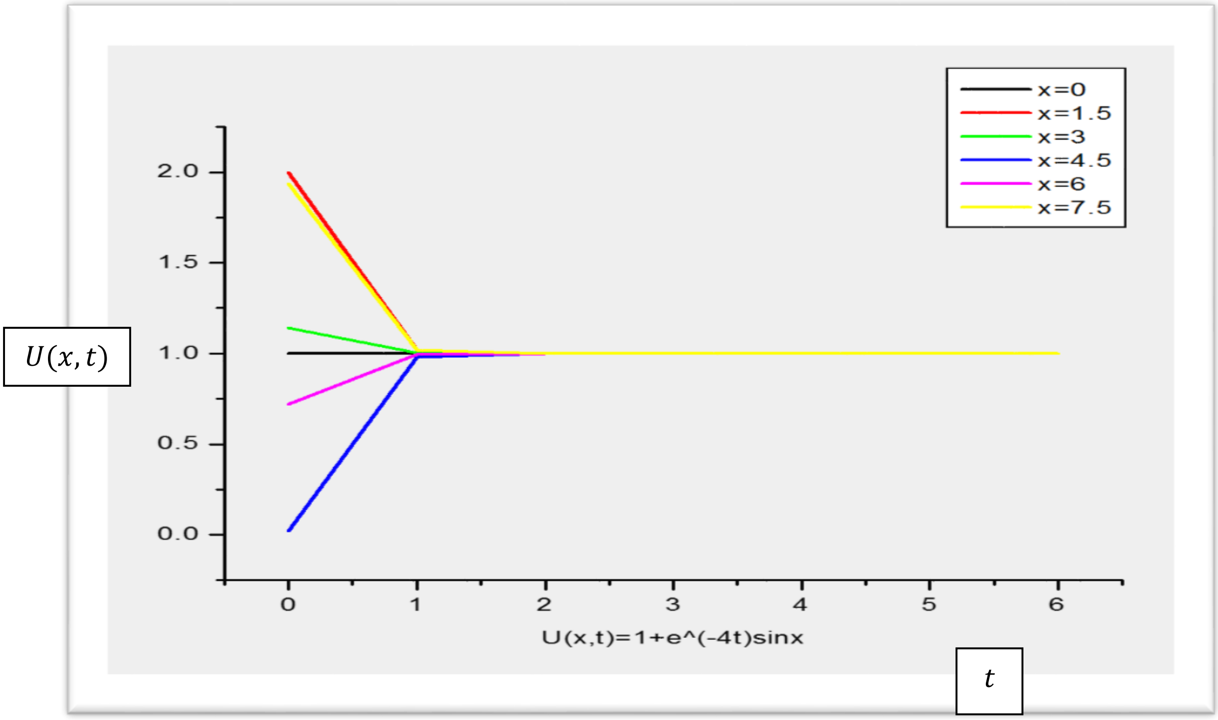


Figure 2.6: Graphical Interpretation of $U(x, t)$ for fix x

Discussion: By observing the numerical values in Table 2.6 and the figure 2.6, that, for fix values of x, the heat flow moves like the exponential wave and moving towards a constant value.

Conclusion

Here, in the present work, we applied the Double Natural transform to find the solution of specific heat equations of one dimension under the boundary conditions. The results obtained here are comparable with the results obtained earlier in the literature with the help of double Laplace Transform and double Sumudu Transform (one can refer [6]). The nature of heat flow can be observed from the figures 2.1, 2.2, 2.3, 2.4 2.5 and 2.6 for the one-dimensional heat equations with different boundary conditions and for different value combinations of x and t which provides a deep thought of application in the relevant field.

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