Applied Mathematics and Computational Intelligence
In Press, Accepted Manuscript – Note to users

A Fitted Numerical Method for a Class of Singularly Perturbed Convection Delayed Dominated Diffusion Equation

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ABSTRACT

A new exponentially fitted numerical method based on uniform mesh is proposed to obtain the solution of a class of singularly perturbed convection delayed dominated diffusion equation. Considered equation is first reduced to the ordinary singularly perturbed problem by expanding the term containing negative shift using Taylor series expansion procedure and then a three term scheme is obtained using the theory of finite differences. A fitting factor is introduced in the derived scheme with the help of singular perturbation theory. Thomas algorithm is employed to find the solution of the resulting tridiagonal system of equations. Stability and convergence of the proposed method are discussed. Method is shown to be first accurate. Computational results for two example problems are presented for different values of grid point, N *and perturbation parameter,* ε. *It is observed that the method is capable of approximating the solution very well.*

Keywords: Differential-difference equations, Exponential fitting factor, Finite difference, Singular perturbations, Stability and Convergence of numerical methods.

1. INTRODUCTION

Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then react to it. The study of theoretical and applied problems in science and engineering leads to the boundary value problems for singularly perturbed differential equations with at least one delay(negative shift) and/or advance(positive shift) term. These equations are of multi scale character in general. That is, there are thin transition layer(s) where solution varies most rapidly while away from the layers(s) solution behaves regularly and varies slowly. These singularly perturbed problems occur very frequently in modeling of various real life applications. Some of the applications of these problems can be found in the high level monographs and the articles: Rihan[1], Stein[2-3], Longtin and Milton[4], Derstine et al.[5], Bocharov and Rihan[6], Nelson and Perelson[7], Mackey and Glass[8], O'Malley [9,16], Knowles and Messick [10], Gold [11], Kaushik [12], Kaushik and Sharma[13], Kadalbajoo and Sharma[14] and Kaushik [15], and Miller [17].

Developing efficient numerical methods for solving such singularly perturbed problems is challenging due to the existence of boundary and/or interior layers. Standard discretization techniques are unable to provide accurate results when the delay and/or advance parameter is/are small/big order of perturbation parameter. Singular perturbation analysis of boundary value problems for differential-difference equation with small shifts was initiated by Lange and Miura [18-20]. In the recent past, various numerical approaches have been suggested for the solution of such problems. The authors, Kadalbajoo and Sharma[21] gave an ε-uniform fitted operator method for solving boundary-value problems for singularly perturbed delay differential equations with Layer behavior. Kadalbajoo and Sharma [22] devised a numerical techniques

Note: Accepted manuscripts are articles that have been peer-reviewed and accepted for publication by the Editorial Board. These articles have not yet been copyedited and/or formatted in the journal house style.

based on finite differences for solving singularly perturbed differential–difference equations. Sharma et al. [23] suggested a way of analytic approximation to delayed convection dominated systems through transforms. Swamy et al.[24] have presented a computational method for singularly perturbed delay differential equations (SPDDE) with twin layers or oscillatory behaviour. Numerical treatment for a singularly perturbed convection delayed dominated diffusion equation via tension splines are presented in Kanth and Kumar[25]. File et al.[26] proposed a fourth order finite difference method for finding the solution of SPDDE. Chakravarthy and Kumar[27], suggested a novel method based on Numerov's difference scheme for the solution of singularly SPDDE of reaction diffusion type. Sirisha and Reddy[28] proposed an exponential fitted numerical integration technique for the solution of SPDDE. An extension of a fitted finite difference method for third order SPDDE of convention diffusion type is presented by the authors Mahendran and Subburayan[29]. Kanth and Kumar[30] proposed a parametric spline scheme for a class of nonlinear SPDDE. The solution of second order SPDDE using Trigonometric B-Spline is suggested by Vaid and Arora [31]. In this paper, we have suggested a new fitted numerical method based on uniform mesh for solving singularly perturbed convection delayed dominated diffusion equation. The paper is arranged as follows: Statement of the considered problems with the description of the proposed methods are presented in Section: 2. Stability and convergence of the method is investigated In Section: 3. Numerical illustrations are given in Section 4. Finally, the discussions and conclusions are presented in the section: 5. Paper ends with the references.

2. PROBLEM STATEMENT

Consider a class of singularly perturbed differential-difference equation of the form:

$$\varepsilon\, y''(x) + a(x)y'(x) + q(x)y(x-\delta) = f(x) \text{ on } \Omega = [0,1], \tag{1}$$

subject to the interval and boundary conditions:

$$y(x) = \eta(x) \text{ on } \delta \le x \le 0, y(1) = \gamma, \tag{2}$$

where $0 < \varepsilon << 1$ is a perturbation parameter and $\delta = o(\varepsilon)$ is a small shifting parameter. Further, the functions $a(x), q(x), f(x)$ and $\eta(x)$ are sufficiently smooth functions in [0, 1] where γ is a constant. The equation (1) reduces to a singularly perturbed ordinary differential equation when $\delta = 0$, and the corresponding problem has a boundary layer on the left side for $a(x) > 0$ or on the right side for $a(x) < 0$ on $[0,1]$.
Using Taylor series expansion for the delayed term, we get from (1);

$$\varepsilon\, y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \text{ on } \Omega = [0,1], \tag{3}$$

subject to the boundary conditions:

$$y(x) = \eta_0 \text{ and } y(1) = \gamma \tag{4}$$

where, $p(x) = \{a(x) - \delta q(x)\}$. Under the assumption that $p(x) \ge M > 0$ throughout the interval $[0,1]$, where M is a positive constant, the problem (3) with (4) exhibits boundary layer at $x = 0$. Similarly when $p(x) \le M < 0$ throughout the interval $[0,1]$, where M is a negative constant, then the problem (3) with (4) exhibits boundary layer at $x = 1$.

Here, the operator $L_\tau = \varepsilon \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)I$ in (3) satisfies the following minimum principle.

Lemma 1. Suppose $\psi(x)$ is a smooth function satisfying $\psi(0)\ge 0, \psi(1)\ge 0$. The $L_\tau\psi(x)\le 0, \forall x\in(0,1)$ implies $\psi(x)\ge 0, \forall x\in[0,1]$.

Proof. We can prove the above lemma by the method of contradiction. Let $k\in[0,1]$ be such that $\psi(k)<0$ and $\psi(k)=\min_{x\in[0,1]}\psi(x)$. Clearly for $k\notin\{0,1\}$, we have $\psi'(k)=0$ and $\psi''(k)\ge 0$. Therefore, we obtain

$$L_\tau\psi(k) = \varepsilon\psi''(k) + p(k)\psi'(k) + q(k)\psi(k) > 0,$$

which is a contradiction to our assumption. Hence it is proved that $\psi(k)\ge 0$ and thus $\psi(x)\ge 0,\ \forall x\in[0,1]$.

Lemma 2. Let $y(x)$ be the solution of the problem (3) and (4) then we have

$$\|y\| \le \theta^{-1}\|f\| + \max(|\eta_0|,|\gamma|). \tag{5}$$

where $\|\cdot\|$ is the L_∞ norm given by $\|y\| = \max_{0\le x\le 1}|y(x)|$.

Proof. Let $\psi^{\pm}(x)$ be two barrier functions define by

$$\psi^{\pm}(x) = \theta^{-1}\|f\| + \max(|\eta_0|,|\gamma|) \pm y(x).$$

Then this implies,

$$\psi^{\pm}(0) = \theta^{-1}\|f\| + \max(|\eta_0|,|\gamma|) \pm y(0) \ = \theta^{-1}\|f\| + \max(|\eta_0|,|\gamma|) \pm \eta_0 \text{ since } y(0)=\eta(0)=\eta_0 \ge 0,$$

$$\psi^{\pm}(1) = \theta^{-1}\|f\| + \max(|\eta_0|,|\gamma|) \pm y(1) \ = \theta^{-1}\|f\| + \max(|\eta_0|,|\gamma|) \pm \gamma \text{ since } y(1)=\gamma \ge 0$$

$$\Rightarrow L_\tau\psi^{\pm}(x) = \varepsilon(\psi^{\pm}(x))'' + p(x)(\psi^{\pm}(x))' + q(x)\psi^{\pm}(x) = q(x)[\theta^{-1}\|f\| + \max(|\eta_0|,|\gamma|)] \pm L_\tau y(x)$$
$$= q(x)\left[\theta^{-1}\|f\| + \max(|\eta_0|,|\gamma|)\right] \pm f(x) \text{ using (3).}$$

As $q(x)\le -\theta < 0$ implies $q(x)\theta^{-1} \le -1$ and since $\|f\| \ge f(x)$, we have

$$L_\tau\psi^{\pm}(x) \le \left(-\|f\| \pm f(x)\right) + q(x)\max(|\eta_0|,|\gamma|) \le 0, \forall x\in[0,1].$$

Thus by the minimum principle we obtain, $\psi^{\pm}\ge 0, \forall x\in[0,1]$, which gives the required estimate.

2.1. Description of the method for left-End Boundary Layer Problems

To describe the method, we consider the singularly perturbed boundary value problem (3)- (4) with $p(x)\ge M > 0$. The theory of singular perturbation gives the solution of (3) with (4) which is of the form (cf.[9], pp.22-26):

$$y(x) = y_0(x) + \frac{p(0)}{p(x)}(\alpha - y_0(0))e^{-\int_0^x\left(\frac{p(x)}{\varepsilon} - \frac{q(x)}{p(x)}\right)dx} + o(\varepsilon), \tag{6}$$

where $y_0(x)$ is the solution of the reduced problem:

$$p(x)y_0'(x)+q(x)y_0(x)=f(x);\ y_0(1)=\beta. \tag{7}$$

Under the consideration of Taylor's series expansions of $p(x)$ and $q(x)$ about the point '$x=0$' upto their first terms only, the equation (6) becomes:

$$y(x)=y_0(x)+\left(\alpha-y_0(0)\right)e^{-\left(\frac{p(0)}{\varepsilon}-\frac{q(0)}{p(0)}\right)x}+o(\varepsilon). \tag{8}$$

Further, considering equation (5) at the point $x=x_i=ih, i=0,1,2,\ldots,N$ and taking the limit as $h\to 0$ we obtain

$$\lim_{h\to 0} y(ih)=y_0(0)+\left(\alpha-y_0(0)\right)e^{-\left(\frac{p^2(0)-\varepsilon q(0)}{p(0)}\right)i\rho}+o(\varepsilon), \tag{9}$$

where $\rho=h/\varepsilon$.

Assuming that $y(x)$ is continuously differentiable in the interval $[0,1]$ and applying Taylor's series expansion for $y(x_{i+1})$ and $y(x_{i-1})$, we have:

$$y(x_{i+1})=y_{i+1}=y_i+hy_i'+\frac{h^2}{2!}y_i''+\frac{h^3}{3!}y_i'''+\frac{h^4}{4!}y_i^{(4)}+\frac{h^5}{5!}y_i^{(5)}+\frac{h^6}{6!}y_i^{(6)}+\frac{h^7}{7!}y_i^{(7)}+\frac{h^8}{8!}y_i^{(8)}+O(h^9) \tag{10}$$

$$y(x_{i-1})=y_{i-1}=y_i-hy_i'+\frac{h^2}{2!}y_i''-\frac{h^3}{3!}y_i'''+\frac{h^4}{4!}y_i^{(4)}-\frac{h^5}{5!}y_i^{(5)}+\frac{h^6}{6!}y_i^{(6)}-\frac{h^7}{7!}y_i^{(7)}+\frac{h^8}{8!}y_i^{(8)}-O(h^9). \tag{11}$$

From finite differences, we have

$$y_{i-1}-2y_i+y_{i+1}=\frac{2h^2}{2!}y_i''+\frac{2h^4}{4!}y_i^{(4)}+\frac{2h^6}{6!}y_i^{(6)}+\frac{2h^8}{8!}y_i^{(8)}+O(h^{10}). \tag{12}$$

Now we have the relation:

$$y_{i-1}''-2y_i''+y_{i+1}''=\frac{2h^2}{2!}y_i^{(4)}+\frac{2h^4}{4!}y_i^{(6)}+\frac{2h^6}{6!}y_i^{(8)}+\frac{2h^8}{8!}y_i^{(10)}+O(h^{12}).$$

Substituting $\frac{h^4}{12}y_i^{(6)}$ from the above equation in (12), we get

$$y_{i-1}-2y_i+y_{i+1}=h^2y_i''+\frac{h^4}{12}y_i^{(4)}+\frac{h^2}{30}\left(y_{i-1}''-2y_i''+y_{i+1}''-h^2y_i^{(4)}-\frac{h^6}{360}y_i^{(8)}\right)+\frac{2h^8}{8!}y_i^{(8)}+O(h^{10})$$

$$y_{i-1}-2y_i+y_{i+1}=h^2\left(y_i''+\frac{1}{30}\left(y_{i-1}''-2y_i''+y_{i+1}''\right)\right)+\frac{h^4}{12}y_i^{(4)}-\frac{h^4}{30}y_i^{(4)}-\frac{h^6}{10800}y_i^{(8)}+\frac{2h^8}{8!}y_i^{(8)}+O(h^{10})$$

$$y_{i-1}-2y_i+y_{i+1}=h^2\left(y_i''+\frac{1}{30}\delta^2y_i''\right)+\frac{h^4}{20}y_i^{(4)}-\frac{13h^6}{302400}y_i^{(8)}+O(h^{10})$$

and

$$y_{i-1}-2y_i+y_{i+1}=\frac{h^2}{30}\left(y_{i-1}''+28y_i''+y_{i+1}''\right)+R, \tag{13}$$

where $R=\frac{h^4}{20}y_i^{(4)}-\frac{13h^6}{302400}y_i^{(8)}+O(h^{10}).$

Now from the equation (3), we have

$$\varepsilon y''_{i+1} = -p_{i+1}y'_{i+1} - q_{i+1}y_{i+1} + f_{i+1} \tag{14}$$

$$\varepsilon y''_i = -p_i y'_i - q_i y_i + f_i \tag{15}$$

$$\varepsilon y''_{i-1} = -p_{i-1}y'_{i-1} - q_{i-1}y_{i-1} + f_{i-1}. \tag{16}$$

Using the following three point approximations for first order derivatives:

$$y'_i = \frac{y_{i+1} - y_{i-1}}{2h} \tag{17}$$

$$y'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} \tag{18}$$

$$y'_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}. \tag{19}$$

Substituting (17), (18) and (19) in (14), (15) and (16) respectively, and simplifying (13), we get

$$\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \frac{p_{i-1}}{60h}(-3y_{i-1} + 4y_i - y_{i+1}) + \frac{28p_i}{60h}(y_{i+1} - y_{i-1}) + \frac{p_{i+1}}{60h}(y_{i-1} - 4y_i + 3y_{i+1}) +$$
$$\frac{q_{i-1}}{30}y_{i-1} + \frac{28q_i}{30}y_i + \frac{q_{i+1}}{30}y_{i+1} = \frac{1}{30}(f_{i-1} + 28f_i + f_{i+1}).$$

Now introducing the fitting factor $\sigma(\rho)$ in the above scheme, we have

$$\sigma(\rho)\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \frac{p_{i-1}}{60h}(-3y_{i-1} + 4y_i - y_{i+1}) + \frac{28p_i}{60h}(y_{i+1} - y_{i-1}) + \frac{p_{i+1}}{60h}(y_{i-1} - 4y_i +$$
$$3y_{i+1}) + \frac{q_{i-1}}{30}y_{i-1} + \frac{28q_i}{30}y_i + \frac{q_{i+1}}{30}y_{i+1} = \frac{1}{30}(f_{i-1} + 28f_i + f_{i+1}). \tag{20}$$

The fitting factor $\sigma(\rho)$ is to be determined in such a way that the solution of difference scheme (20) converges uniformly to the solution of equation (3) with (2) and hence (1) with (2).

Multiplying (20) by h and taking the limit as $h \to 0$, we get

$$\sigma(\rho)\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \frac{p(0)}{60}(-3y_{i-1} + 4y_i - y_{i+1}) + \frac{28p(0)}{60}(y_{i+1} - y_{i-1}) +$$
$$\frac{p(0)}{60}(y_{i-1} - 4y_i + 3y_{i+1}) = 0. \tag{21}$$

Let $P = \frac{p^2(0) - \varepsilon q(0)}{p(0)}$. By using (9), we get

$$\lim_{h\to 0}\left(y(ih-h) - 2y(ih) + y(ih+h)\right) = (\alpha - y_0(0))e^{-Pi\rho}\left(e^{P\rho} + e^{-P\rho} - 2\right)$$

$$\lim_{h\to 0}\left(-3y(ih-h) + 4y(ih) - y(ih+h)\right) = (\alpha - y_0(0))e^{-Pi\rho}\left(-3e^{P\rho} - e^{-P\rho} + 4\right)$$

$$\lim_{h\to 0}\left(y(ih-h) - 4y(ih) + 3y(ih+h)\right) = (\alpha - y_0(0))e^{-Pi\rho}\left(e^{P\rho} + 3e^{-P\rho} - 4\right) \text{ and}$$

$$\lim_{h\to 0}\left(y(ih+h)-y(ih-h)\right)=\left(\alpha-y_0(0)\right)e^{-Pi\rho}\left(e^{-P\rho}-e^{P\rho}\right).$$

By using the above equations in equation (21), we get

$$\frac{\sigma(\rho)}{\rho}\left(e^{P\rho}+e^{-P\rho}-2\right)=-\frac{p(0)}{60}\left(-30e^{P\rho}+30e^{-P\rho}\right).$$

Therefore $$\sigma(\rho)=\frac{\rho p(0)}{2}\coth\left(\frac{\left(p^2(0)-\varepsilon q(0)\right)\rho}{2p(0)}\right), \quad (22)$$

which is a required constant fitting factor $\sigma(\rho)$.

Finally, by making use of equation (20) and $\sigma(\rho)$ given by equation (22), we get the following three-term recurrence relationship:

$$E_i y_{i-1}-F_i y_i+G_i y_{i+1}=R_i,\ (i=1,2,3,\ldots,N-1), \quad (23)$$

where

$$E_i=\frac{\sigma\varepsilon}{h^2}-\frac{3p_{i-1}}{60h}+\frac{q_{i-1}}{30}-\frac{28p_i}{60h}+\frac{p_{i+1}}{60h}$$
$$F_i=\frac{2\sigma\varepsilon}{h^2}-\frac{4p_{i-1}}{60h}-\frac{28q_{i-1}}{30}+\frac{4p_{i+1}}{60h}$$
$$G_i=\frac{\sigma\varepsilon}{h^2}-\frac{p_{i-1}}{60h}+\frac{q_{i+1}}{30}+\frac{28p_i}{60h}+\frac{3p_{i+1}}{60h}$$
$$R_i=\frac{1}{30}\left(f_{i-1}+28f_i+f_{i+1}\right)$$

Equation (23) gives a system of $(N-1)$ equations with $(N-1)$ unknowns y_1 to y_{N-1}. These $(N-1)$ equations together with the Eq. (4) are sufficient to solve the obtained tri-diagonal system by using an efficient solver called 'Thomas Algorithm' described in [32].

We assume the matrix of this set of linear equations as D_N.

Lemma: For all $\varepsilon>0$ and all $h=1/N$, the matrix D_N is an irreducible and diagonally dominant matrix.

Proof. Clearly, D_N is a tri-diagonal matrix. Hence, D_N is irreducible if its co-diagonals contain non zero elements only. It is easily seen that the co-diagonals E_i, G_i do not vanish for all $\varepsilon>0$, $h>0$ and $\alpha_i\in R$. Hence D_N is irreducible.

Since E_i, G_i do not vanish for all $\varepsilon>0$, $h>0$ and $\alpha_i\in R$ these expressions are of constant sign. Then obviously, $E_i>0$, $G_i>0$.

Now in each row of D_N the modulus of the diagonal element is greater than or equal to the sum of the two off-diagonal elements for all $i,(i=1,2,3,\ldots,N-1)$. This proves that D_N is diagonally dominant. Under the above mentioned conditions the 'Thomas Algorithm' is stable. The method of LU decomposition (or Gaussian elimination) which is equivalent to the "Thomas algorithm" provides a numerically stable technique for solving the system when the coefficient matrix of the system is diagonally dominant or irreducibly diagonally dominant and hence non singular.

2.1.1. Description of the method for Right-End Boundary Layer Problems

To describe the method, we again consider the singularly perturbed boundary value problem (3) - (4) with $p(x) \le M < 0$. The theory of singular perturbation provides the solution of (3) with (4) in the form (cf.[9], pp.22-26):

$$y(x) = y_0(x) + \frac{p(1)}{p(x)}\left(\beta - y_0(1)\right)e^{\int_x^1\left(\frac{p(x)}{\varepsilon} - \frac{q(x)}{p(x)}\right)dx} + o(\varepsilon), \tag{24}$$

where $y_0(x)$ represents the solution of the reduced problem:

$$p(x)y_0'(x) + q(x)y_0(x) = f(x);\ y_0(0) = \alpha. \tag{25}$$

Expanding $p(x)$ and $q(x)$ in (24) with the help of the Taylor's series about the point '$x=1$' and restricting to their first terms, we obtain:

$$y(x) = y_0(x) + \left(\beta - y_0(1)\right)e^{\left(\frac{p(1)}{\varepsilon} - \frac{q(1)}{p(1)}\right)(1-x)} + o(\varepsilon). \tag{26}$$

Further, considering equation (26) at the point $x = x_i = ih, i = 0,1,2,\ldots,N$ and taking the limit as $h \to 0$ we obtain

$$\lim_{h\to 0} y(ih) = y_0(0) + \left(\beta - y_0(1)\right)e^{\left(\frac{p^2(1)-\varepsilon q(1)}{p(1)}\right)\left(\frac{1}{\varepsilon} - i\rho\right)} + o(\varepsilon), \tag{27}$$

where $\rho = h/\varepsilon$.

Let $\hat{P} = \dfrac{p^2(1) - \varepsilon q(1)}{p(1)}$. By using (9), we get

$$\lim_{h\to 0}\left(y(ih-h) - 2y(ih) + y(ih+h)\right) = \left(\beta - y_0(1)\right)e^{\hat{P}\left(\frac{1}{\varepsilon} - i\rho\right)}\left(e^{\hat{P}\rho} + e^{-\hat{P}\rho} - 2\right)$$

$$\lim_{h\to 0}\left(-3y(ih-h) + 4y(ih) - y(ih+h)\right) = \left(\beta - y_0(1)\right)e^{\hat{P}\left(\frac{1}{\varepsilon} - i\rho\right)}\left(-3e^{\hat{P}\rho} - e^{-\hat{P}\rho} + 4\right)$$

$$\lim_{h\to 0}\left(y(ih-h) - 4y(ih) + 3y(ih+h)\right) = \left(\beta - y_0(0)\right)e^{\hat{P}\left(\frac{1}{\varepsilon} - i\rho\right)}\left(e^{\hat{P}\rho} + 3e^{-\hat{P}\rho} - 4\right)$$

$$\lim_{h\to 0}\left(y(ih+h) - y(ih-h)\right) = \left(\alpha - y_0(0)\right)e^{\hat{P}\left(\frac{1}{\varepsilon} - i\rho\right)}\left(e^{-\hat{P}\rho} - e^{\hat{P}\rho}\right).$$

By using the above equations in equation (21), we get

$$\frac{\sigma(\rho)}{\rho}\left(e^{\hat{P}\rho} + e^{-\hat{P}\rho} - 2\right) = -\frac{p(0)}{60}\left(-30e^{\hat{P}\rho} + 30e^{-\hat{P}\rho}\right)$$

Therefore
$$\sigma(\rho) = \frac{\rho p(0)}{2}\coth\left(\frac{\left(p^2(1) - \varepsilon q(1)\right)\rho}{2p(1)}\right). \tag{28}$$

which is a required constant fitting factor. Using equation (20) with $\sigma(\rho)$ given by equation (28), we have the three-term recurrence relationship (23).

3. STABILITY AND CONVERGENCE ANALYSIS

Theorem 1: Assuming the conditions $\varepsilon > 0$, $p(x) > M > 0$ and $q(x) < 0, \forall x \in [0,1]$, the system of difference equations (23), along with the given boundary conditions, possesses a solution which is unique and satisfies

$$\|y\|_{h,\infty} \le 2M^{-1}\|R\|_{h,\infty} + (|\eta| + |\gamma|),$$

where $\|.\|_{h,\infty}$ denotes the discrete $l_\infty -$ norm, given by $\|x\|_{h,\infty} = \max_{0\le i\le N}\{|x_i|\}$.

Proof: Let the difference operator given on the left hand side of (14) is denoted by $L_h(.)$ and ω_i represents any mesh function which satisfies $L_h(\omega_i) = f_i$. Rearranging the difference scheme (23) and using the condition that the coefficients of E_i, F_i and G_i are non-negative, we obtain

$$F_i|\omega_i| \le |R_i| + E_i|\omega_{i-1}| + G_i|\omega_{i+1}|$$

$$\Rightarrow \left(\frac{2\sigma\varepsilon}{h^2} - \frac{4p_{i-1}}{60h} - \frac{28q_{i-1}}{30} + \frac{4p_{i+1}}{60h}\right)|\omega_i| \le |R_i| + \left(\frac{\sigma\varepsilon}{h^2} - \frac{3p_{i-1}}{60h} + \frac{q_{i-1}}{30} - \frac{28p_i}{60h} + \frac{p_{i+1}}{60h}\right)|\omega_{i-1}| + \left(\frac{\sigma\varepsilon}{h^2} - \frac{p_{i-1}}{60h} + \frac{q_{i+1}}{30} + \frac{28p_i}{60h} + \frac{3p_{i+1}}{60h}\right)|\omega_{i+1}|$$

$$\Rightarrow \sigma\varepsilon\left(\frac{|\omega_{i+1}| - 2|\omega_i| + |\omega_{i-1}|}{h^2}\right) + \frac{p_{i-1}}{30}\left(\frac{-|\omega_{i+1}| + 4|\omega_i| - 3|\omega_{i-1}|}{2h}\right) + \frac{q_{i-1}}{30}|\omega_{i-1}| + \frac{28q_i}{30}|\omega_i| + \frac{q_{i+1}}{30}|\omega_{i+1}| + \frac{28p_i}{30}\left(\frac{|\omega_{i+1}| - |\omega_{i-1}|}{2h}\right) + \frac{p_{i+1}}{30}\left(\frac{3|\omega_{i+1}| - 4|\omega_i| + |\omega_{i-1}|}{2h}\right) + |R_i| \ge 0.$$

Now, taking help of the assumptions $\varepsilon > 0$ and $p(x) > M$ and using the definition of $l_\infty -$ norm the above inequality gets changed into

$$\sigma\varepsilon\left(\frac{|\omega_{i+1}| - 2|\omega_i| + |\omega_{i-1}|}{h^2}\right) + \frac{M}{30}\left(\frac{-|\omega_{i+1}| + 4|\omega_i| - 3|\omega_{i-1}|}{2h}\right) + \frac{q_{i-1}}{30}|\omega_{i-1}| + \frac{28q_i}{30}|\omega_i| + \frac{q_{i+1}}{30}|\omega_{i+1}| + \frac{28M}{30}\left(\frac{|\omega_{i+1}| - |\omega_{i-1}|}{2h}\right) + \frac{M}{30}\left(\frac{3|\omega_{i+1}| - 4|\omega_i| + |\omega_{i-1}|}{2h}\right) + |R_i| \ge 0. \quad (29)$$

To prove the uniqueness and existence of the solution, let $\{u_i\},\{v_i\}$ represents two sets of solution of the difference equation (29) which satisfies the boundary conditions. Then $\omega_i = u_i - v_i$ satisfies the condition $L_h(\omega_i) = R_i$ where $R_i = 0$ and $\omega_0 = \omega_N = 0$. Performing summation over $i = 1, 2, \ldots, N-1$, in (29), we obtain

$$\sum_{i=1}^{N-1}\sigma\varepsilon\left(\frac{|\omega_{i+1}|-2|\omega_i|+|\omega_{i-1}|}{h^2}\right)+\sum_{i=1}^{N-1}\frac{M}{30}\left(\frac{-|\omega_{i+1}|+4|\omega_i|-3|\omega_{i-1}|}{2h}\right)+\sum_{i=1}^{N-1}\frac{q_{i-1}}{30}|\omega_{i-1}|+\sum_{i=1}^{N-1}\frac{28q_i}{30}|\omega_i|+$$
$$\sum_{i=1}^{N-1}\frac{q_{i+1}}{30}|\omega_{i+1}|+\sum_{i=1}^{N-1}\frac{28M}{30}\left(\frac{|\omega_{i+1}|-|\omega_{i-1}|}{2h}\right)+\sum_{i=1}^{N-1}\frac{M}{30}\left(\frac{3|\omega_{i+1}|-4|\omega_i|+|\omega_{i-1}|}{2h}\right)+\sum_{i=1}^{N-1}|R_i|\geq 0$$

$$\Rightarrow -\sigma\varepsilon\frac{|\omega_1|}{h^2}-\sigma\varepsilon\frac{|\omega_{N-1}|}{h^2}+\|p\|_{h,\infty}\frac{|\omega_1|}{60h}+\|p\|_{h,\infty}\frac{3|\omega_{N-1}|}{60h}+\frac{1}{30}\sum_{i=1}^{N-1}q_{i-1}|\omega_{i-1}|+\frac{28}{30}\sum_{i=1}^{N-1}q_i|\omega_i|+$$
$$\frac{1}{30}\sum_{i=1}^{N-1}q_{i+1}|\omega_{i+1}|-\frac{28M}{60h}|\omega_1|-\frac{28M}{60h}|\omega_{N-2}|-\frac{3M}{60h}|\omega_1|-\frac{M}{60h}|\omega_{N-1}|+\sum_{i=1}^{N-1}|R_i|\geq 0. \quad (30)$$

Since, $\varepsilon>0,\|p\|_{h,\infty}\geq 0, q_i<0$ and $|\omega_i|\geq 0, \forall i, i=1,2,\dots,N-1,$ therefore for the inequality (30) to be true, we must have

$$\omega_i=0\,\forall i, i=1,2,\dots,N-1.$$

This implies that the solution of the tri-diagonal system of difference equations (23) is unique. In case of linear equations, the uniqueness of the solution implies its existence.

Now to validate the estimate, let $\omega_i=y_i-l_i,$ where y_i satisfies the difference equations (23) along with the boundary conditions and $l_i=(1-ih)\eta+(ih)\gamma,$ then $\omega_0=\omega_N=0$ and $\omega_i, i=1,2,\dots,N-1.$

Now, let $|\omega_n|=\|\omega\|_{h,\infty}\geq|\omega_i|, i=0,1,2,\dots,N.$ Then performing summation over (29) from $i=n$ to $N-1$ and taking the help of the assumption on $p(x),$ we obtain

$$-\sigma\varepsilon\left(\frac{|\omega_n|-|\omega_{n-1}|}{h^2}\right)-\sigma\varepsilon\frac{|\omega_{N-1}|}{h^2}+\frac{M}{30}\left(\frac{-|\omega_{N-1}|+4|\omega_n|-3|\omega_{n-1}|}{2h}\right)+\frac{1}{30}\sum_{i=n}^{N-1}q_{i-1}|\omega_{i-1}|+\frac{28}{30}\sum_{i=n}^{N-1}q_i|\omega_i|+$$
$$\frac{1}{30}\sum_{i=n}^{N-1}q_{i+1}|\omega_{i+1}|+\frac{28M}{30}\left(\frac{|\omega_{N-1}|-|\omega_n|-|\omega_{n-1}|}{2h}\right)+\frac{M}{30}\left(\frac{-|\omega_{N-1}|-3|\omega_n|-|\omega_{n-1}|}{2h}\right)+\sum_{i=n}^{N-1}|R_i|\geq 0 \quad (31)$$

Taking the condition on $q(x)$ into consideration, inequality (28, 31) implies

$$\frac{M}{2}|\omega_n|\leq h\sum_{i=n}^{N-1}|R_i|\leq h\sum_{i=0}^{N}|h_i|\leq\|R\|_{h,\infty},$$

i.e we have

$$|\omega_n|\leq 2M^{-1}\|R\|_{h,\infty} \quad (32)$$

Again, we have $y_i=\omega_i+l_i$

$$\|y\|_{h,\infty}=\max_{0\leq i\leq N}\{|y_i|\}\leq\|\omega\|_{h,\infty}+\|l\|_{h,\infty}$$
$$\leq\|\omega_n\|+\|l\|_{h,\infty}. \quad (33)$$

Now to complete the proof of the estimate, the bound on l_i has to be found.

$$\|l\|_{h,\infty} = \max_{0\le i\le N}\{|l_i|\} \le \max_{0\le i\le N}\{|(1-ih)||\eta| + |ih||\gamma|\}$$
$$\le \max_{0\le i\le N}\{(1-ih)|\eta| + (ih)|\gamma|\}$$

i.e., we have

$$\|l\|_{h,\infty} \le |\eta| + |\gamma| \tag{34}$$

From equations (33)-(34), we obtain the estimate

$$\|y\|_{h,\infty} \le 2M^{-1}\|R\|_{h,\infty} + (|\eta| + |\gamma|).$$

This theorem establishes the fact that the solution to the system of the difference equations (23) are uniformly bounded and is independent of the mesh size h and perturbation parameter ε. This proves the stability of the scheme for all step sizes.

Corollary 1: Under the conditions stated in theorem-1, the error $e_i = y(x_i) - y_i$, which occurs between $y(x)$, the solution of the continuous problem (3)-(4) and y_i, the solution of the discretized problem, together with boundary conditions, satisfies the estimate

$$\|e\|_{h,\infty} = 2M^{-1}\|\tau\|_{h,\infty},$$

Where $|\tau_i| \le \max_{x_{i-1}\le i\le x_{i+1}}\left\{\frac{\sigma h^2\varepsilon}{12}|y^{(4)}(x)|\right\} + \max_{x_{i-1}\le i\le x_{i+1}}\left\{\frac{28ph^2}{180}|y^{(3)}(x)|\right\}$

(The coefficients $p(x), q(x)$ are assumed to be sufficiently differentiable to ensure that the solution $y(x)$ belongs to $C^3[0,1]$).

Proof: Truncation error in the difference scheme, denoted by τ_i, is written as

$$\tau_i = \sigma\varepsilon\left\{\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) - y_i''\right\} + \frac{p_{i-1}}{30}\left\{\left(\frac{-3y_{i-1} + 4y_i - y_{i+1}}{2h}\right) - y_{i-1}'\right\} + \frac{28p_i}{30}\left\{\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) - y_i'\right\} +$$
$$\frac{p_{i+1}}{30}\left\{\left(\frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h}\right) - y_{i+1}'\right\}.$$

$$\Rightarrow \tau_i = \sigma\varepsilon\left\{\frac{h^2}{12}y_i^{(4)} + \frac{h^4}{360}y_i^{(6)} + \dots\right\} + \frac{p_{i-1}}{30}\left\{-\frac{h^2}{3}y_i^{(3)} - \frac{h^3}{12}y_i^{(4)} + \dots\right\} + \frac{28p_i}{30}\left\{\frac{h^2}{6}y_i^{(3)} + \frac{h^4}{120}y_i^{(5)} + \dots\right\} +$$
$$\frac{p_{i+1}}{30}\left\{-\frac{h^2}{3}y_i^{(3)} - \frac{h^3}{12}y_i^{(4)} + \dots\right\}$$

$$\Rightarrow |\tau_i| \le \max_{x_{i-1}\le i\le x_{i+1}}\left\{\frac{\sigma h^2\varepsilon}{12}|y^{(4)}(x)|\right\} - \max_{x_{i-1}\le i\le x_{i+1}}\left\{\frac{ph^3}{90}|y^{(3)}(x)|\right\} + \max_{x_{i-1}\le i\le x_{i+1}}\left\{\frac{28ph^2}{180}|y^{(3)}(x)|\right\} + \max_{x_{i-1}\le i\le x_{i+1}}\left\{\frac{ph^3}{90}|y^{(3)}(x)|\right\}$$

$$\Rightarrow |\tau_i| \le \max_{x_{i-1}\le i\le x_{i+1}}\left\{\frac{\sigma h^2\varepsilon}{12}|y^{(4)}(x)|\right\} + \max_{x_{i-1}\le i\le x_{i+1}}\left\{\frac{28ph^2}{180}|y^{(3)}(x)|\right\} \tag{35}$$

It can be easily shown that the error e_i satisfies

$$L_h(e(x_i)) = L_h(y(x_i)) - L_h(y_i) = \tau_i, i = 1, 2, \dots, N-1$$

and $$e_0 = e_N = 0.$$

Then Theorem-1 implies that

$$\|e\|_{h,\infty} \le 2M^{-1}\|\tau\|_{h,\infty} \tag{36}$$

For fixed values of the parameter ε, the convergence of the difference scheme is established by the estimate (36).

Theorem 2: Under the assumptions $\varepsilon > 0,\ p(x) \le M < 0$ and $q(x) < 0, \forall x \in [0,1]$, the system of difference equations (23), along with the given boundary conditions, possesses a solution which is unique and satisfies

$$\|y\|_{h,\infty} \le 2M^{-1}\|R\|_{h,\infty} + \left(|\eta| + |\gamma|\right)$$

where $\|.\|_{h,\infty}$ represents the discrete $l_\infty -$ norm, given by $\|x\|_{h,\infty} = \max_{0\le i\le N}\{|x_i|\}$.

The proof of estimate can be established in a similar manner as done in theorem 1.

4. NUMERICAL ILLUSTRATIONS

In order to show the effectiveness of the proposed method, we have solved the two test example problems and presented the computational results in terms of maximum absolute errors. These test problems have been chosen because they have been widely discussed in the literature and the approximate solutions are available for comparison. The computational results are presented in the Tables 1 to 6 for various values of grid point N and perturbation parameter ε.

Note that the maximum absolute errors (MAE): E_ε^N are calculated by $E_\varepsilon^N = \max_{0\le i\le N}\left[|y(x_i) - y_i|\right]$, where $y(x_i)$ and y_i denote the exact and approximate solution respectively. Since the exact solutions of the problems are not known, the maximum absolute errors for the examples are calculated using the following double mesh principle $E_\varepsilon^N = \max_{0\le i\le N}\left[\left|y_i^N - y_i^{2N}\right|\right]$. The computational rate of convergence is obtained by using the double mesh principle defined below:

Let $Z_h = \max\left|y_j^h - y_j^{h/2}\right|,\ j = 1,\ldots,N-1,$ where y_j^h is the computed solution on the mesh $\{x_j\}_0^N$ at the nodal point x_j, where $x_j = x_{j-1} + h,\ j = 1,2,\ldots,N$ and $y_j^{h/2}$ is the computed solution at the nodal point x_j on the mesh $\{x_j\}_0^{2N}$, where $x_j = x_{j-1} + h/2,\ j = 1,2,\ldots,2N.$ In the same way we can define $Z_{h/2}$ by replacing h by $h/2$ and N by $2N$ i.e., $Z_{h/2} = \max\left|y_j^{h/2} - y_j^{h/4}\right|,\ j = 1,\ldots,2N-1.$

The computed rate of convergence is defined as $R_\varepsilon^N = \dfrac{\log Z_h - \log Z_{h/2}}{\log(2)}.$

We have taken $h = 2^{-5}$ for finding the computed rate of convergence for the example problems and the results are shown in the Table 7 and Table 8. Clearly, the computational rate of convergence agrees with the theoretical rate of convergence and the proposed method gives a second order convergent.

Problem 1: First we consider the following constant coefficient convection delayed dominated diffusion equation:

$$\varepsilon y''(x)+5y'(x)+y(x-\delta)=0; x\in[0,1]$$

with interval boundary conditions $y(x)=1$ on $-\delta\le x\le 0$ and $y(1)=0$, which has a boundary layer at the left side of the domain near $x=0$.

The exact solution of this example problem is given by: $y(x)=\dfrac{\gamma-\eta e^{m_2}}{e^{m_1}-e^{m_2}}e^{m_1x}+\dfrac{\eta e^{m_1}-\gamma}{e^{m_1}-e^{m_2}}e^{m_2x}$

where $m_1=\dfrac{-(p-\delta q)+\sqrt{(p-\delta q)^2-4\varepsilon q}}{2\varepsilon}, m_2=\dfrac{-(p-\delta q)-\sqrt{(p-\delta q)^2-4\varepsilon q}}{2\varepsilon}$.

The computational results in terms of maximum absolute errors for the Problem 1 is given in Tables 1, 2 and 3 for $\delta=0.1$, $\delta=0.3*\varepsilon$ and $\delta=0.5*\varepsilon$ respectively.

Table 1 Maximum absolute errors for different values of N and ε with $\delta=0.1$ for the problem-1.

$\varepsilon\downarrow$	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
2^{-2}	3.02393E-03	4.03771E-03	3.87004E-03	3.88572E-03	3.89066E-03	3.88840E-03
2^{-3}	1.05762E-03	1.48334E-03	1.99890E-03	1.91614E-03	1.92806E-03	1.92919E-03
2^{-4}	8.73248E-04	5.21208E-04	7.34664E-04	9.94653E-04	9.53466E-04	9.60916E-04
2^{-5}	8.71867E-04	4.31150E-04	2.58763E-04	3.65622E-04	4.96119E-04	4.75496E-04
2^{-6}	8.71870E-04	4.30487E-04	2.14248E-04	1.28916E-04	1.82368E-04	2.47687E-04
2^{-7}	8.71870E-04	4.30490E-04	2.13911E-04	1.06780E-04	6.43576E-05	9.10759E-05
2^{-8}	8.71870E-04	4.30490E-04	2.13914E-04	1.06628E-04	5.33440E-05	3.21586E-05
2^{-9}	8.71870E-04	4.30490E-04	2.13914E-04	1.06631E-04	5.32351E-05	2.66634E-05
2^{-10}	8.71870E-04	4.30490E-04	2.13914E-04	1.06631E-04	5.32382E-05	2.66004E-05
2^{-12}	8.71870E-04	4.30490E-04	2.13914E-04	1.06631E-04	5.32382E-05	2.66035E-05
2^{-16}	8.71870E-04	4.30490E-04	2.13914E-04	1.06631E-04	5.32382E-05	2.66035E-05
2^{-32}	8.71870E-04	4.30490E-04	2.13914E-04	1.06631E-04	5.32382E-05	2.66035E-05

Table 2 Maximum absolute errors for different values of N and ε with $\delta=0.3*\varepsilon$ for the problem-1.

$\varepsilon\downarrow$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-1}	1.47474E-03	1.48383E-03	1.48931E-03	1.50031E-03	1.49804E-03	1.49792E-03
10^{-2}	2.09726E-04	1.05763E-04	8.17031E-05	1.30370E-04	1.52379E-04	1.48565E-04

10^{-3}	2.09620E-04	1.04496E-04	5.21722E-05	2.60666E-05	1.30188E-05	7.88132E-06
10^{-4}	2.09597E-04	1.04479E-04	5.21584E-05	2.60574E-05	1.30218E-05	6.50761E-06
10^{-5}	2.09566E-04	1.04449E-04	5.21283E-05	2.60276E-05	1.29920E-05	6.47786E-06
10^{-6}	2.09582E-04	1.04465E-04	5.21444E-05	2.60437E-05	1.30081E-05	6.49396E-06
10^{-7}	2.09603E-04	1.04485E-04	5.21651E-05	2.60644E-05	1.30288E-05	6.51465E-06
10^{-8}	2.09600E-04	1.04483E-04	5.21624E-05	2.60617E-05	1.30261E-05	6.51195E-06
10^{-9}	2.09600E-04	1.04482E-04	5.21622E-05	2.60614E-05	1.30258E-05	6.51168E-06
10^{-10}	2.09600E-04	1.04482E-04	5.21621E-05	2.60614E-05	1.30258E-05	6.51165E-06
10^{-12}	2.09600E-04	1.04482E-04	5.21621E-05	2.60614E-05	1.30258E-05	6.51165E-06
10^{-15}	2.09600E-04	1.04482E-04	5.21621E-05	2.60614E-05	1.30258E-05	6.51165E-06
10^{-20}	2.09600E-04	1.04482E-04	5.21621E-05	2.60614E-05	1.30258E-05	6.51165E-06
10^{-30}	2.09600E-04	1.04482E-04	5.21621E-05	2.60614E-05	1.30258E-05	6.51165E-06

Table 3 Maximum absolute errors for different values of N and ε with $\delta = 0.5*\varepsilon$ for the problem-1.

$\varepsilon \downarrow$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-1}	1.48876E-03	1.49423E-03	1.50222E-03	1.51142E-03	1.51065E-03	1.51050E-03
10^{-2}	2.09824E-04	1.05759E-04	8.18148E-05	1.30460E-04	1.52498E-04	1.48654E-04
10^{-3}	2.09601E-04	1.04473E-04	5.21471E-05	2.60404E-05	1.30197E-05	7.89575E-06
10^{-4}	2.09595E-04	1.04477E-04	5.21558E-05	2.60548E-05	1.30191E-05	6.50492E-06
10^{-5}	2.09576E-04	1.04458E-04	5.21376E-05	2.60368E-05	1.30012E-05	6.48713E-06
10^{-6}	2.09602E-04	1.04485E-04	5.21645E-05	2.60637E-05	1.30281E-05	6.51396E-06
10^{-7}	2.09605E-04	1.04487E-04	5.21671E-05	2.60664E-05	1.30308E-05	6.51665E-06

10^{-8}	2.09600E-04	1.04483E-04	5.21626E-05	2.60619E-05	1.30263E-05	6.51215E-06
10^{-9}	2.09600E-04	1.04482E-04	5.21622E-05	2.60614E-05	1.30258E-05	6.51170E-06
10^{-10}	2.09600E-04	1.04482E-04	5.21621E-05	2.60614E-05	1.30258E-05	6.51165E-06
10^{-12}	2.09600E-04	1.04482E-04	5.21621E-05	2.60614E-05	1.30258E-05	6.51165E-06
10^{-15}	2.09600E-04	1.04482E-04	5.21621E-05	2.60614E-05	1.30258E-05	6.51165E-06
10^{-20}	2.09600E-04	1.04482E-04	5.21621E-05	2.60614E-05	1.30258E-05	6.51165E-06
10^{-30}	2.09600E-04	1.04482E-04	5.21621E-05	2.60614E-05	1.30258E-05	6.51165E-06

Problem 2: Now, we consider the following constant coefficient convection delayed dominated diffusion equation from:

$$\varepsilon y''(x) - 5y'(x) + 2y(x-\delta) = 0; x \in [0,1]$$

with interval boundary conditions $y(x)=0$ on $-\delta \le x \le 0$ and $y(1)=2,$ which has a boundary layer at the left side of the domain near $x=1.$ The exact solution of this example is not given. The computational results in terms of maximum absolute errors for the example Problem 2 is given in Tables 4, 5 and 6 for $\delta=0.1,$ $\delta=0.3*\varepsilon$ and $\delta=0.5*\varepsilon$ respectively.

Table 4 Maximum absolute errors for different values of N and ε with $\delta=0.1$ for problem- 2.

$\varepsilon \downarrow$	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
2^{-3}	3.10985E-03	1.17642E-03	3.35574E-04	7.40886E-05	1.65105E-05	1.00136E-05
2^{-4}	3.34127E-03	1.51792E-03	5.79402E-04	1.65939E-04	3.82066E-05	7.27177E-06
2^{-5}	3.36014E-03	1.63122E-03	7.50063E-04	2.87622E-04	8.26120E-05	1.81198E-05
2^{-6}	3.36014E-03	1.64030E-03	7.50063E-04	3.72832E-04	1.43260E-04	4.13656E-05
2^{-7}	3.36024E-03	1.64036E-03	8.10560E-04	4.00718E-04	1.85880E-04	7.14660E-05
2^{-8}	3.36024E-03	1.64036E-03	8.10586E-04	4.02911E-04	1.99774E-04	9.28054E-05
2^{-9}	3.36024E-03	1.64036E-03	8.10586E-04	4.02925E-04	2.00854E-04	9.97361E-05
2^{-10}	3.36024E-03	1.64036E-03	8.10586E-04	4.02925E-04	2.00862E-04	1.00262E-04
2^{-12}	3.36024E-03	1.64036E-03	8.10586E-04	4.02925E-04	2.00862E-04	1.00267E-04
2^{-16}	3.36024E-03	1.64036E-03	8.10586E-04	4.02925E-04	2.00862E-04	1.00267E-04
2^{-32}	3.36024E-03	1.64036E-03	8.10586E-04	4.02925E-04	2.00862E-04	1.00267E-04

Table 5 Maximum absolute errors for different values of N and ε and $\delta=0.3*\varepsilon$ for the problem-2.

$\varepsilon\downarrow$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-1}	4.16249E-04	9.45330E-05	2.49743E-05	5.78165E-06	3.27826E-05	3.99351E-06
10^{-2}	8.42048E-04	4.10273E-04	1.78698E-04	6.16014E-05	1.53184E-05	4.23193E-06
10^{-3}	8.43312E-04	4.19100E-04	2.08906E-04	1.04270E-04	5.16622E-05	2.38419E-05
10^{-4}	8.43449E-04	4.19189E-04	2.08973E-04	1.04338E-04	5.21380E-05	2.60674E-05
10^{-5}	8.43385E-04	4.19122E-04	2.08903E-04	1.04267E-04	5.20667E-05	2.59958E-05
10^{-6}	8.43418E-04	4.19153E-04	2.08934E-04	1.04298E-04	5.20977E-05	2.60268E-05
10^{-7}	8.43431E-04	4.19166E-04	2.08947E-04	1.04311E-04	5.21104E-05	2.60394E-05
10^{-8}	8.43441E-04	4.19177E-04	2.08947E-04	1.04322E-04	5.21212E-05	2.60502E-05
10^{-9}	8.43443E-04	4.19178E-04	2.08959E-04	1.04323E-04	5.21222E-05	2.60513E-05
10^{-10}	8.43443E-04	4.19178E-04	2.08959E-04	1.04323E-04	5.21224E-05	2.60514E-05
10^{-12}	8.43443E-04	4.19178E-04	2.08959E-04	1.04323E-04	5.21224E-05	2.60514E-05
10^{-15}	8.43443E-04	4.19178E-04	2.08959E-04	1.04323E-04	5.21224E-05	2.60514E-05
10^{-20}	8.43443E-04	4.19178E-04	2.08959E-04	1.04323E-04	5.21224E-05	2.60514E-05
10^{-30}	8.43443E-04	4.19178E-04	2.08959E-04	1.04323E-04	5.21224E-05	2.60514E-05

Table 6 Computational results in terms of Maximum absolute errors for different values of N and ε and $\delta=0.5*\varepsilon$ for the problem- 2.

$\varepsilon\downarrow$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-1}	4.15653E-04	9.51290E-05	2.48551E-05	5.72205E-06	3.13520E-05	7.21216E-06
10^{-2}	8.41315E-04	4.09924E-04	1.78579E-04	6.16610E-05	1.53780E-05	2.98023E-06
10^{-3}	8.43257E-04	4.19079E-04	2.08903E-04	1.04275E-04	5.16589E-05	2.36854E-05
10^{-4}	8.43357E-04	4.19101E-04	2.08887E-04	1.04253E-04	5.20531E-05	2.59827E-05
10^{-5}	8.43444E-04	4.19180E-04	2.08962E-04	1.04325E-04	5.21250E-05	2.60541E-05

10^{-6}	8.43433E-04	4.19169E-04	2.08950E-04	1.04314E-04	5.21131E-05	2.60421E-05
10^{-7}	8.43422E-04	4.19158E-04	2.08939E-04	1.04303E-04	5.21023E-05	2.60314E-05
10^{-8}	8.43441E-04	4.19176E-04	2.08957E-04	1.04321E-04	5.21204E-05	2.60494E-05
10^{-9}	8.43442E-04	4.19178E-04	2.08959E-04	1.04323E-04	5.21222E-05	2.60512E-05
10^{-10}	8.43443E-04	4.19178E-04	2.08959E-04	1.04323E-04	5.21223E-05	2.60514E-05
10^{-12}	8.43443E-04	4.19178E-04	2.08959E-04	1.04323E-04	5.21224E-05	2.60514E-05
10^{-15}	8.43443E-04	4.19178E-04	2.08959E-04	1.04323E-04	5.21224E-05	2.60514E-05
10^{-20}	8.43443E-04	4.19178E-04	2.08959E-04	1.04323E-04	5.21224E-05	2.60514E-05
10^{-30}	8.43443E-04	4.19178E-04	2.08959E-04	1.04323E-04	5.21224E-05	2.60514E-05

Table 7 The Rate of convergence R_ε^N for various values of N and ε and $\delta = 0.5*\varepsilon$ for the problem 1.

$\varepsilon \downarrow$	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^{-3}	7.294E-07	1.779E-07	4.393E-08	1.091E-08	2.719E-09	6.783E-10	1.666E-10
R_ε^N	2.0356	2.0178	2.0096	2.0045	2.0031	2.0255	
10^{-4}	7.293E-07	1.779E-07	4.393E-08	1.092E-08	2.720E-09	6.789E-10	1.695E-10
R_ε^N	2.0354	2.0178	2.0082	2.0053	2.0023	2.0019	
10^{-5}	7.293E-07	1.779E-07	4.392E-08	1.091E-08	2.718E-09	6.779E-10	1.690E-10
R_ε^N	2.0354	2.0181	2.0092	2.0050	2.0034	2.0040	
10^{-6}	7.293E-07	1.779E-07	4.393E-08	1.092E-08	2.721E-09	6.793E-10	1.697E-10
R_ε^N	2.0354	2.0178	2.0082	2.0048	2.0020	2.0011	
10^{-7}	7.293E-07	1.779E-07	4.393E-08	1.092E-08	2.721E-09	6.795E-10	1.698E-10
R_ε^N	2.0354	2.0178	2.0082	2.0048	2.0016	2.0006	
10^{-8}	7.293E-07	1.779E-07	4.393E-08	1.092E-08	2.721E-09	6.792E-10	1.697E-10
R_ε^N	2.0354	2.0178	2.0082	2.0048	2.0022	2.0008	
10^{-9}	7.293E-07	1.779E-07	4.393E-08	1.092E-08	2.721E-09	6.792E-10	1.697E-10
R_ε^N	2.0354	2.0178	2.0082	2.0048	2.0022	2.0008	
10^{-10}	7.293E-07	1.779E-07	4.393E-08	1.092E-08	2.721E-09	6.792E-10	1.697E-10
R_ε^N	2.0354	2.0178	2.0082	2.0048	2.0022	2.0008	

10^{-15} R_ε^N	7.293E-07 2.0354	1.779E-07 2.0178	4.393E-08 2.0082	1.092E-08 2.0048	2.721E-09 2.0022	6.792E-10 2.0008	1.697E-10

Table 8 The Rate of convergence R_ε^N for various values of N and ε and $\delta = 0.3*\varepsilon$ for the problem 2.

$\varepsilon \downarrow$	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^{-3} R_ε^N	6.134E-06 2.0728	1.458E-06 2.0353	3.557E-07 2.0179	8.783E-08 2.0091	2.182E-08 2.0045	5.438E-09 0.4222	7.287E-09
10^{-4} R_ε^N	6.135E-06 2.0721	1.459E-06 2.0358	3.558E-07 2.0176	8.787E-08 2.0084	2.184E-08 2.0045	5.443E-09 2.0019	1.359E-09
10^{-5} R_ε^N	6.135E-06 2.0731	1.458E-06 2.0353	3.557E-07 2.0177	8.784E-08 2.0092	2.182E-08 2.0050	5.436E-09 2.0043	1.355E-09
10^{-6} R_ε^N	6.135E-06 2.0721	1.459E-06 2.0358	3.558E-07 2.0180	8.785E-08 2.0087	2.183E-08 2.0049	5.439E-09 2.0029	1.357E-09
10^{-7} R_ε^N	6.135E-06 2.0721	1.459E-06 2.0358	3.558E-07 2.0178	8.786E-08 2.0089	2.183E-08 2.0044	5.441E-09 2.0024	1.358E-09
10^{-8} R_ε^N	6.135E-06 2.0721	1.459E-06 2.0358	3.558E-07 2.0178	8.786E-08 2.0089	2.183E-08 2.0041	5.442E-09 2.0027	1.358E-09
10^{-9} R_ε^N	6.135E-06 2.0721	1.459E-06 2.0358	3.558E-07 2.0178	8.786E-08 2.0089	2.183E-08 2.0041	5.442E-09 2.0027	1.358E-09
10^{-10} R_ε^N	6.135E-06 2.0721	1.459E-06 2.0358	3.558E-07 2.0178	8.786E-08 2.0089	2.183E-08 2.0041	5.442E-09 2.0027	1.358E-09
10^{-15} R_ε^N	6.135E-06 2.0721	1.459E-06 2.0358	3.558E-07 2.0178	8.786E-08 2.0089	2.183E-08 2.0041	5.442E-09 2.0027	1.358E-09

5. DISCUSSIONS AND CONCLUSIONS

We have examined a three-point exponentially fitted second order numerical method for singularly perturbed delayed boundary value problem having boundary layer at one end of the considered region. Introduction of a fitting factor in a second order finite difference scheme is made to take care of the rapid changes occurring in the boundary layer. The existence and uniqueness of the discrete problem along with stability estimates are discussed. We have presented maximum absolute errors for the standard examples chosen from the literature and also presented maximum absolute errors for some of the examples to show the efficiency of the method when $\varepsilon << N$. The computational results are presented in Tables 1-6. It is easily observed that the MAE for the problem is becoming uniform, when singular perturbation parameter $\varepsilon \to 0$, for any fixed value of $N = \frac{1}{h}$. If the value of δ is small, for any fixed value of $N = \frac{1}{h}$, the maximum absolute error decreases. Tables show that the method presented in this paper is capable of producing second order accurate results on uniform mesh with minimal computational effort. The main feature of the proposed fitted scheme is that it does not depends on the very fine mess size like Shishkin mesh.

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