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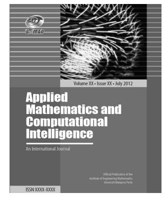

A numerical study of the mathematical model of flow in the petroleum reservoirs

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Abstract: *This paper concerns with the solution of a nonlinear, degenerate, convection-diffusion problem describing two-phase flow in porous media. A numerical procedure based on decomposition scheme is developed to solve the proposed problem. For illustration purpose, two test problems are considered and their series and exact solutions are compared.*

Keywords: *Petroleum reservoirs, porous media, diffusion equation, Adomian decomposition method.*

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1 Introduction

Motivation for the following mathematical problem arises from the area of modelling flow and transport of contaminants in groundwater and petroleum reservoirs. Flow simulation in porous media has been extensively studied using finite element methods in past years (see, e.g., [1-3] and the bibliographies therein). Also, discretizations using both finite element and finite volume methods for two-phase flow in porous media are presented in [2, 3].

Petroleum reservoir and groundwater aquifer simulation often requires the solution of a nonlinear, degenerate, convection-diffusion problem describing two-phase flow in porous media (see, e.g. [1-4]). In this paper, we will focus on immiscible flow, which corresponds physically to water flooding of a petroleum reservoir. We consider two-phase water and oil flow in a porous media, using the global pressure, the total velocity and the water saturation as the primary variables.

Diffusion equation is one of the most important models which appears in porous media, and often is nonlinear. Nonlinear partial differential equations are encountered in such various fields as physics, mathematics and engineering. Most nonlinear models of real life

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problems are still very difficult to solve either numerically or theoretically. There has recently been much attention devoted to the search for better and more efficient methods for determining a solution, approximate or exact, analytical or numerical, to the nonlinear models [4-12]. The objective of this paper is to present a method based on Adomian decomposition method (ADM) for solving the following nonlinear, degenerate, convection-diffusion equation with appropriate initial and boundary conditions

$$\phi(x)u_t + div(b(u)\overrightarrow{q}) - div(K(x)\nabla a(u)) = 0, \quad (x,t) \in Q_T = \Omega \times J, \tag{1}$$

where $u(x,t)$ is the water saturation, $\overrightarrow{q}$ is the total velocity, $\phi(x)$ is the porosity of the porous medium and $K(x)$ is the absolute permeability tensor of the reservoir Ω and $a(u)$ and $b(u)$ are nonlinear functions which depend on the mobilities and the capillary pressure with diffusion coefficient vanishing for two values of saturation: $a(0) = a(1) = 0$ (degeneration of the diffusion term).

2 Adomian decomposition method (ADM)

The ADM has been proved to be effective and reliable for handling differential equations, linear or nonlinear. Unlike the traditional methods, The ADM needs no discritization, linearization, spatial transformation or perturbation. The ADM provide an analytical solution in the form of an infinite convergent power series. A large amount of research works has been devoted to the application of the ADM to a wide class of linear and nonlinear, ordinary or partial differential equations [6, 8-14].

Let us first recall the basic principles of the ADM for solving differential equations. Consider the general equation: $\Psi u = g$, where Ψ represents a general nonlinear differential operator involving both linear and nonlinear terms. The linear term is decomposed into $L + R$, where L is easily invertible and R is the remainder of the linear operator. For convenience, L may be taken as the highest order derivation. Thus the equation may be written as

$$Lu + Ru + Nu = q, \tag{2}$$

where Nu represents the nonlinear terms. Solving Lu from (2), we have

$$Lu = g - Ru - Nu. \tag{3}$$

Since L is invertible, the equivalent expression is

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{4}$$

Therefor, u can be expressed as following series

$$u = \sum_{n=0}^{\infty} u_n, \tag{5}$$

with reasonable u_0 which may be identified with respect to the definition of L^{-1} and g, and $u_n,\ n > 0$ is to be determined. The nonlinear term Nu will be decomposed by the infinite series of Adomian polynomials

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where A_n's are obtained by writing

$$v(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n, \tag{7}$$

$$N(v(\lambda)) = \sum_{n=0}^{\infty} \lambda^n A_n. \tag{8}$$

Here λ is a parameter introduced for convenience. From (7) and (8), we have

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{k=0}^{\infty}\lambda^k u_k)]_{\lambda=0}, \quad n \geq 0. \tag{9}$$

The A_n's are given as

$$\begin{aligned}
A_0 &= F(u_0),\\
A_1 &= u_1\frac{d}{du_0}F(u_0),\\
A_2 &= u_2\frac{d}{du_0}F(u_0) + \frac{u_1^2}{2!}\frac{d^2}{du_0^2}F(u_0),\\
A_3 &= u_3\frac{d}{du_0}F(u_0) + u_1u_2\frac{d^2}{du_0^2}F(u_0) + \frac{u_1^3}{3!}\frac{d^3}{du_0^3}F(u_0),\\
&\vdots
\end{aligned}$$

Now, substituting (5) and (6) into (4) yields

$$\sum_{n=0}^{\infty}u_n = u_0 + L^{-1}R(\sum_{n=0}^{\infty}u_n) - L^{-1}\sum_{n=0}^{\infty}A_n. \tag{10}$$

Consequently, with a suitable u_0 we can write

$$\begin{aligned}
u_1 &= -L^{-1}Ru_0 - L^{-1}A_0,\\
&\vdots\\
u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n.
\end{aligned}$$

All of u_n are calculable, and $u = \sum_{n=0}^{\infty}u_n$. Since the series converges and does so very rapidly, the n-term partial sum $S_n = \sum_{k=0}^{n-1}\lambda^k u_k$ can serve as a practical solution.

For the convergence of the decomposition method, the readers are referred to [8, 11, 14].

3 Problem definition and numerical solution

Consider following one dimensional nonlinear, degenerate, convection-diffusion problem

$$\begin{aligned}
\phi(x)\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}(K(x)\frac{\partial}{\partial x}a(u)) - \frac{\partial}{\partial x}(q(x)b(u)),\\
&\quad (x,t) \in (0,1)\times(0,T) &(11)\\
u(x,0) &= f(x),\ x \in (0,1), &(12)\\
u(0,t) &= \varphi_1(t),\ t \in (0,T), &(13)\\
u(1,t) &= \varphi_2(t),\ t \in (0,T), &(14)\\
0 \leq\ u(x,t) &\leq 1 &(15)
\end{aligned}$$

where $\phi(x)$ and $K(x)$ are positive functions.

In this section, we consider following linear operators

$$L_{xx} = \frac{\partial^2}{\partial x^2}, \quad L_x = \frac{\partial}{\partial x}, \quad L_t = \frac{\partial}{\partial t}. \tag{16}$$

Using this notation, the equation (11) becomes

$$\phi(x)L_t(u) + L_x(q(x)b(u)) - L_x(K(x)L_x a(u)) = 0. \tag{17}$$

By defining the inverse operators L_t^{-1} one may formally obtain from (4) that

$$u(x,t) = f(x) + \frac{1}{\phi(x)}(L_t^{-1}[L_x(K(x)L_x(a(u))) - L_x(q(x)b(u))]), \tag{18}$$

where $L_t^{-1}(.) = \int_0^t (.)d\tau$. Now if we define

$$u_0(x,t) = f(x), \tag{19}$$

we can seek the solution $u(x,t)$ of the problem (11)-(15) based on Adomian decomposition approach as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{20}$$

The nonlinear terms are decomposed as

$$a(u) = \sum_{n=0}^{\infty} A_n, \tag{21}$$

$$b(u) = \sum_{n=0}^{\infty} B_n, \tag{22}$$

where A_n and B_n may be found by using Adomian polynomials. Substituting (20), (21) and (22) into (18) gives

$$\sum_{n=0}^{\infty} u_n = f(x) + \frac{1}{\phi(x)}(L_t^{-1}[L_x(K(x)L_x(\sum_{n=0}^{\infty} A_n) + L_x(q(x)\sum_{n=0}^{\infty} B_n)]). \tag{23}$$

Using above decomposition analysis, the following recurrence relation can be derived

$$u_0 = f(x) \tag{24}$$

$$u_{n+1} = \frac{1}{\phi(x)}(L_t^{-1}[L_x(K(x)L_x A_n) - L_x(q(x)B_n)]), \quad n \geq 0. \tag{25}$$

On the other hand, we can use the operator L_{xx} and it's inverse to represent the solution. Using (13) and (14) the operator L_{xx}^{-1} and the starting term may be derived as

$$L_{xx}^{-1} = \int_0^x dx' \int_0^{x'} dx'' - x\int_0^1 dx' \int_0^{x'} dx'' \tag{26}$$

$$\tilde{u}_0 = (1-x)\varphi_1(t) + x\varphi_2(t). \tag{27}$$

Then the ADM yields following series solution for the problem (11)-(15)

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{28}$$

where for $n \geq 0$

$$u_{n+1} = L_{xx}^{-1}[\frac{1}{K(x)}\{L_t u_n - L_x(K(x)L_x A_n) - K^{'}(x)L_x u_n + L_x(q(x)B_n)\}]. \tag{29}$$

Table 1: Decomposition solutions S_6.

x	t	Exact	S6	Absolute errors
0.2	0.3	5.00242026E-4	5.00133504E-4	1.08521533E-7
	0.6	5.00484052E-4	5.00334801E-4	1.49250462E-7
	0.9	5.00726077E-4	5.00536098E-4	1.89979346E-7
0.4	0.3	5.00272026E-4	5.00128692E-4	1.43334091E-7
	0.6	5.00544052E-4	5.00356134E-4	1.87917588E-7
	0.9	5.00816077E-4	5.00583576E-4	2.32501015E-7
0.6	0.3	5.00302026E-4	5.00164199E-4	1.37826021E-7
	0.6	5.00604057E-4	5.00417787E-4	1.86264116E-7
	0.9	5.00906077E-4	5.00671375E-4	2.34702119E-7

In (29), $K^{'}(x)$ shows the derivation of $K(x)$ with respect to variable x and the Adomian polynomial A_n is obtained using the function $a(u) - u$. The decomposition series (25) and (29) are generally convergent very rapidly in real physical problems. One can use each one of the decomposition series (25) or (29) for constructing the solution of the problem (11)-(15). In addition if one wants to introduce the solution with respect to the initial and boundary conditions (12)-(14), the average of the relations (25) or (29) can be used.

4 Test problems

In this section, for illustrations purpose we consider some problems and we show that how the ADM presented in the preceding section is computationally efficient.

Example 1. Consider following nonlinear initial-boundary value problem

$$\begin{cases} u_t = u_{xx} + u(1-u)(u-10^{-3}), \quad 0<x<1,\ 0<t<1 \\ u(x,0) = \frac{1}{2}(1+\tanh(x)) \\ u(0,t) = \frac{1}{2}(1+\tanh(\frac{\sqrt{2}}{2}(2-10^{-3})t)) \\ u(1,t) = \frac{1}{2}(1+\tanh\{1+\frac{\sqrt{2}}{2}(2-10^{-3})t\}). \end{cases} \tag{30}$$

The exact solution of this problem can be derived as [14]

$$u(x,t) = \frac{1}{2}(1+\tanh\{x+\frac{\sqrt{2}}{2}(2-10^{-3})t\}). \tag{31}$$

Table 1 shows decomposition solution using $S_6(x,t)$, exact solution $u(x,t)$, and the absolute errors between them at some points.

Example 2. Consider the following initial-boundary value problem

$$u_t = u_{xx} + uu_x + \frac{1}{9}u(u^2-36), \quad 0<x<1,\ 0<t<1 \tag{32}$$

$$u(x,0) = \frac{6(e^{2x}-1)}{1+e^x+e^{2x}}, \quad 0<x<1 \tag{33}$$

$$u(0,t) = 0, \quad 0<t<1 \tag{34}$$

$$u_x(0,t) = \frac{12}{2+e^{3t}}, \quad 0<t<1. \tag{35}$$

The exact solution of problem this problem is

$$u(x,t) = \frac{6(e^{2x}-1)}{1+e^{2x}+e^{x+3t}}. \tag{36}$$

Table 2: Decomposition solutions S_6.

x	t	Exact	S_6	Absolute errors
0.2	0.3	0.536927	0.536928	5.67744E-7
	0.6	0.298652	0.298653	5.85133E-7
	0.9	0.142793	0.142795	1.37153E-7
0.4	0.3	1.06649	1.06648	6.81273E-7
	0.6	0.600238	0.600239	7.81525E-7
	0.9	0.289230	0.289231	1.14918E-7
0.6	0.3	1.58157	1.58159	6.67887E-7
	0.6	0.907283	0.907284	9.80515E-7
	0.9	0.442872	0.442873	5.17242E-7

Table 2 shows decomposition solution using $S_6(x,t)$, exact solution $u(x,t)$, and the absolute errors between them at some points.

5 Conclusion

The decomposition method for numerically solving the one dimensional convection-diffusion equation in groundwater and petroleum reservoirs has been established in this paper. This method has the advantages that it needs no discritization, linearization, spatial transformation or perturbation and it seems that this method is a reasonable method for solving the nonlinear problems.

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